A Non-classification Result for Wild Knots

Vadim Kulikov*

April 22, 2016

Using methods of descriptive theory it is shown that the classification problem for wild knots is strictly harder than that for countable structures.

1 Introduction

A knot is a homeomorphic image of S^1 in S^3. Two knots k and k' are equivalent if there exists a homeomorphism $h\colon S^3 \to S^3$ such that $h[k] = k'$, where $f[A]$ denotes the image of the set A under f. Denote this equivalence relation by E_K. If a knot is equivalent to a finite polygon, then it is *tame* and otherwise *wild*. There has been a significant effort to classify various subclasses of wild knots: from early [FH62, DH67, Lom64, McP73] till recent [JL09, Fri03, Nan14]. Also higher dimensional wild knots have been addressed [Fri05, Fri04], and replacing S^1 by the Cantor set one obtains the related study of the so-called wild Cantor sets whose classification (up to an analogously defined equivalence relation) has also been of interest because of their importance in dynamical systems [GRWŽ11].

In this paper it is shown that E_K does not admit classification by countable structures while the isomorphism on the latter is classifiable by E_K. More precisely we show that E_K is strictly above the isomorphism relation on countable structures in the Borel reducibility hierarchy. This is proved by Borel reducing both the isomorphism on linear orders (which is sufficient by a theorem of L. Stanley and H. Friedman, cf. Theorem 2.4 of this paper) and a turbulent equivalence relation to E_K (Theorems 3.1 and 4.1).

This implies in particular that wild knots cannot be completely classified[1] by real numbers considered up to any Borel equivalence relation (see remark after Theorem 2.4).

*Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Straße 25, 1090, Vienna Austria

[1]That is, not in any *reasonable* way, cf. [Gao08, Hjo00, Ros11] for the theory of classification complexity. An example of a *non-reasonable* classification would be to attach real numbers to the knot types using the Axiom of Choice.

Acknowledgements I would like to thank Rami Luisto without whom the proof of Theorem 4.1 would have been considerably more cumbersome, if existed at all. The research was made possible by the Austrian Science Fund (FWF) under project number P24654 and was also partially supported by the Finnish Academy Grant WBS1285203.

2 Preliminaries

In this Section we introduce the basic notions and theorems that we need in the paper.

Polish Spaces and Borel Reductions

For any metric space (X, d) we adopt the following notations:

$$\begin{array}{rcl} B_X(x,\delta) & = & \{y \in X \mid d(x,y) < \delta\}, \\ \bar{B}_X(x,\delta) & = & \{y \in X \mid d(x,y) \leqslant \delta\}, \\ B_X(A,\delta) & = & \{y \in X \mid d(A,y) < \delta\}, \\ \bar{B}_X(A,\delta) & = & \{y \in X \mid d(A,y) \leqslant \delta\}, \end{array}$$

where $d(A, y) = \inf\{d(x, y) \mid x \in A\}$. We drop "$X$" from the lower case when it is obvious from the context.

2.1 Definition. A *Polish* space is a separable topological space which is homeomorphic to a complete metric space. The collection of Borel subsets of a Polish space is the smallest σ-algebra containing the basic open sets. A *Polish* group is a topological group which is a Polish space and both the group operation and the inverse are continuous functions.

A pair $(A, \mathcal{B})$ is a *standard Borel space* if $\mathcal{B}$ is a σ-algebra on A and there exists a Polish topology on A for which $\mathcal{B}$ is precisely the collection of Borel sets.

A function $f: A \to B$, where A and B are standard Borel spaces, is *Borel*, if the inverse image of every Borel set is Borel.

Suppose A and B are standard Borel spaces. Then an equivalence relation $E \subset A \times A$ is *Borel reducible* to an equivalence relation $E' \subset B \times B$, if there exists a Borel map $f: A \to B$ such that for all $x, y \in A$ we have

$$xEy \iff f(x)E'f(y).$$

In this case we can also say that the elements of A considered up to E-equivalence are classified in a Borel way by the elements of B considered up to E'-equivalence, or just denote it by $E \leqslant_B E'$. This is a quasiorder (transitive and reflexive) on equivalence relations.

If G is a Polish group which acts in a Borel way on a standard Borel space B through an action π, then we denote by $E^B_{G,\pi}$ the *orbit equivalence relation* defined by

$$(x, y) \in E^B_{G,\pi} \iff \exists g \in G(\pi(g, x) = y).$$

In this case we say that $E^B_{G,\pi}$ is *induced* by the action of G on B and often drop "π" from the notation when it is either irrelevant or clear from the context. Such an equivalence relation is analytic as a subset of $B \times B$.

2.2 Fact. *([Kec94, Cor 13.4]) If $(A, \mathcal{B})$ is a standard Borel space and $B \in \mathcal{B}$, then $(B, \mathcal{B} \restriction B)$ is also a standard Borel space.* $\square$

2.3 Example. The complex numbers $\mathbb{C}$ is a Polish space and the unit circle $S^1 \subset \mathbb{C}$ is a Polish group whose group operation is the multiplication of complex numbers. The countable product $(S^1)^{\mathbb{N}}$ is a Polish group as well in the product topology. Given a compact space X, the group of self-homeomorphisms $\mathrm{Hom}(X)$ is Polish in the topology induced by the sup-metric [Kec94].

Let L be any countable (first-order) vocabulary and let $S(L)$ be the set of all countable L-structures with domain $\mathbb{N}$. This space can be viewed as a Polish space, in fact homeomorphic to the Cantor space $2^{\mathbb{N}}$. For instance, if L contains just one binary relation R, then for each $\eta \in 2^{\mathbb{N}}$, let $\mathcal{A}_\eta$ be the model with domain $\mathbb{N}$ such that for all $x, y \in \mathbb{N}$, $R^{\mathcal{A}_\eta}(x, y) \iff \eta(\pi(x, y)) = 1$, where $\pi \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection. Denote by $\cong_{S(L)}$ the isomorphism relation $S(L)$ and more generally, if $M \subset S(L)$ is closed under isomorphism, denote $\cong_M = \cong_{S(L)}\restriction M$. It is not hard to see that $\cong_{S(L)}$ is induced by an action of the infinite symmetric group S_∞ (cf. [Gao08]) which is Polish as the subspace of the Baire space $\mathbb{N}^{\mathbb{N}}$.

Let $L = \{<\}$ be a vocabulary consisting of one binary relation and let $\mathrm{LO} \subset S(L)$ be the set of linear orders. It is a closed subset [FS89] and so Polish itself.

2.4 Theorem (H. Friedman and L. Stanley [FS89])**.** *$\cong_{\mathrm{LO}}$ is $\leqslant_B$-maximal among all isomorphism relations, i.e. $\cong_{S(L)} \leqslant_B \cong_{\mathrm{LO}}$ holds for any vocabulary L.* $\square$

In particular $\cong_{\mathrm{LO}}$ is not Borel as a subset of $\mathrm{LO} \times \mathrm{LO}$, because the isomorphism relation on graphs is complete analytic as a set [FS89, Thm 4] and is Borel reducible to $\cong_{\mathrm{LO}}$ by the above. If an equivalence relation E is reducible to $\cong_{S(L)}$ for some L (or equivalently to $\cong_{\mathrm{LO}}$) we say, following [Hjo00], that E *admits classification by countable structures.*

Turbulence

Turbulent group actions were introduced by G. Hjorth in [Hjo00]. The idea of turbulence is roughly to characterize those Polish group actions whose orbit equivalence relation does not admit classification by countable structures.

2.5 Definition (Hjorth)**.** Suppose that X is a Polish space and let G be a Polish group acting continuously on X. For $x \in X$ denote by $[x] = [x]_G$ the orbit of x. This action is said to be *turbulent*, if the following conditions are satisfied:

1. every orbit is dense,

2. every orbit is meager,

3. for every $x, y \in X$, every open $U \subset X$ with $x \in U$ and every open $V \subset G$ with $1_G \in V$, there exist a $y_0 \in [y]_G$ and sequences $(g_i)_{i\in\mathbb{N}}$ in V and $(x_i)_{i\in\mathbb{N}}$ in U such that y_0 is an accumulation point of the set $\{x_i \mid i \in \mathbb{N}\}$ and $x_0 = x$ and $x_{i+1} = g_i x_i$ for all $i \in \mathbb{N}$.

Equivalence relations arising from turbulent actions do not admit classification by countable structures [Hjo00, Cor 3.19]:

2.6 Theorem (Hjorth)**.** *Suppose that a Polish group G acts on a Polish space X in a turbulent way. Then E_G^X does not admit classification by countable structures, i.e. $E_G^X \not\leqslant_B \cong_{S(L)}$ for any L.* □

By Theorems 2.4 and 2.6, if one wants to show that some equivalence relation E is strictly $\leqslant_B$-above the isomorphism of countable structures, it is sufficient to prove $\cong_{\mathrm{LO}} \leqslant_B E$ and $E^* \leqslant_B E$ for some turbulent E^*. This is our plan for $E = E_K$ In this case the turbulent equivalence relation E^* will be a certain relation on $(S^1)^{\mathbb{N}}$.

2.7 Definition. Let us represent the unit circle as $S^1 = [-\pi, \pi]/\{-\pi, \pi\}$, i.e. the closed interval $[-\pi, \pi] \subset \mathbb{R}$ with the end-points identified. Denote by $S^1_{\mathbb{C}}$ the unit circle in the complex plane equipped with the group structure induced from multiplication in $\mathbb{C}$ and let $\tau\colon S^1_{\mathbb{C}} \to S^1$ be the homeomorphism $\tau\colon e^{i\theta} \mapsto \theta$. This induces canonically a group structure on S^1.

Let $X = (S^1)^{\mathbb{N}}$ be the set of sequences on S^1 with the Tychonoff product topology. Since S^1 is a topological group, this induces a group structure on X. Let $G \subset X$ be the subgroup of those $\bar{x}$ for which $\lim_{n\to\infty} x_n = 0$. This group is Polish when equipped with the sup-metric which induces a finer topology than that inherited from the ambient space, but gives the same Borel sets, i.e. the group is *Polishable*. We say that two sequences $\bar{x}', \bar{x} \in (S^1)^{\mathbb{N}}$ are E^*-equivalent if $\bar{x}' - \bar{x} \in G$. Thus E^* is induced by the action of G given by translation. If one replaces S^1 by $\mathbb{R}$ in the above, then this equivalence relation is proved to be turbulent in [Hjo00, § 3.3]. The same proof works in this case as well; the only difference is that in his proof Hjorth uses division of reals by natural numbers and taking their absolute value. The only problem is the element $\bar{\pi} = \{-\pi, \pi\}$ of S^1, so for the sake of the proof one can define $\bar{\pi}/k = \pi/k$ for all $k \geqslant 2$ and $|\bar{\pi}| = \pi$ and otherwise simply compute everything as in $\mathbb{R}$. The essential properties that $|x/k| \xrightarrow{k\to\infty} 0$ and $(\underbrace{x/k + \cdots + x/k}_{k\ times}) = x$ are then preserved.

The Space of Knots

We think of S^3 as the one-point compactification $\mathbb{R}^3 \cup \{\infty\}$. The unit circle S^1 is parametrized as described in Definition 2.7, but this parametrization is relevant only in Section 4. Let $K(S^3)$ be the space of compact subsets of S^3 equipped with the Hausdorff metric:

$$d_{K(S^3)}(K_0, K_1) = \inf\{\varepsilon \mid K_0 \subset B(K_1, \varepsilon) \wedge K_1 \subset B(K_0, \varepsilon)\}. \tag{1}$$

This space is compact Polish [Kec94]. Let $\mathcal{K}$ be the subset of $K(S^3)$ consisting of knots, i.e. homeomorphic images of S^1. The following is a consequence of a more general theorem by Ryll-Nardzewski [RN65]:

2.8 Theorem. *$\mathcal{K}$ is a Borel subset of $K(S^3)$ and so by Fact 2.2 it is a standard Borel space.* □

2.9 Definition. Suppose X, Y are a topological spaces and $A \subset X, B \subset Y$ are subsets. We say that the pairs (X, A) and (Y, B) are homeomorphic (as pairs) if there exists a homeomorphism $h\colon X \to Y$ with $h[A] = B$. Thus, a knot can be thought of as a pair (S^3, K) where K is homeomorphic to S^1. Two knots K and K' are equivalent, if (S^3, K) and (S^3, K') are homeomorphic as pairs. This is the equivalence relation on $\mathcal{K}$ denoted by E_K.

E_K is induced by a Polish group action in a natural way. Let $\operatorname{Hom}(S^3)$ be the group of homeomorphisms of S^3 onto itself. It is a Polish group (cf. Example 2.3) and it acts on $\mathcal{K}$ by $h \cdot K = h[K]$.

2.10 Definition. Let I be a set homeomorphic to the unit interval $[0, 1]$ and let $\bar{B}$ be a set homeomorphic to a closed ball in S^3. We say that $f\colon I \to \bar{B}$ or $(\bar{B}, \operatorname{Im} f)$ or $(\bar{B}, f)$ is a *proper arc in $\bar{B}$* if $x \in \partial I \iff f(x) \in \partial\bar{B}$. Two proper arcs $f\colon I \to \bar{B}$, $f'\colon I \to \bar{B}'$ are equivalent if $(\bar{B}, \operatorname{Im} f)$ and $(\bar{B}', \operatorname{Im} f')$ are homeomorphic as pairs. In the case of knots and proper arcs we often abuse notation by denoting $\operatorname{Im} f$ by f. A proper arc is *tame* if it is equivalent to a smooth arc or equivalently a finite polygon. It is *unknotted* or *trivial* if it is equivalent to

$$(\bar{B}_{\mathbb{R}^3}(\bar{0}, 1), [-1, 1] \times \{(0, 0)\}).$$

We say that a proper arc $(\bar{B}, f)$ has $(\bar{B}', g)$ as *as a component*, if there exists a proper arc $(\bar{B}_0, h)$ and $\bar{B}_1 \subset \bar{B}_0$ such that $(\bar{B}, f)$ is equivalent to $(\bar{B}_0, h)$, $(\bar{B}_1, h \cap \bar{B}_1)$ is a proper arc and $(\bar{B}', g)$ is equivalent to $(\bar{B}_1, h \cap \bar{B}_1)$. This is clearly independent of the choice of representatives.

3 Embedding $\cong_{\mathrm{LO}}$ into E_K

3.1 Theorem. *There is a Borel reduction $\cong_{\mathrm{LO}} \leqslant_B E_K$.*

Proof. The idea is the following. A linear order L is embedded into the unit interval I such that every point in the image is isolated. Then the knot is constructed by replacing a small neighborhood of each such point by a singular knot depicted on Figure 1. Now the order type of the isolated singularities of the resulting knot is the same as the order type of L. All the technical details below are there to ensure that this function is indeed a Borel reduction of $\cong_{\mathrm{LO}}$ into E_K.

Similarly as in Definition 2.3, let LO be the set of those $R \in 2^{\mathbb{N}\times\mathbb{N}}$ which define a linear order on $\mathbb{N}$. Let LO^* be the space of those $(R, S) \in (2^{\mathbb{N}\times\mathbb{N}})^2$ for which R defines a linear order on $\mathbb{N}$ with 0 being the smallest element and with no greatest element

and $S(n,m)=1$ if and only if there are no elements R-between n and m. Since S is first-order definable from R, there is a canonical Borel map $\mathrm{LO}\to\mathrm{LO}^*$ which is a reduction from $\cong_{\mathrm{LO}}$ to $\cong_{\mathrm{LO}^*}$: For a linear order L, first let $L'=1+L+1+1+\mathbb{Q}$ and then canonically define (R,S) so that R is L'. Obviously this is a Borel reduction and so it is sufficient to reduce $\cong_{\mathrm{LO}^*}$ into E_K. Suppose $L=(R_L,S_L)$ is an element of LO^*, where we also denote $R_L={<_L}$. Let $f_L\colon\mathbb{N}\to\mathbb{R}$ be a map and (V_n^L) a sequence of open intervals defined by induction as follows. First $f_L(0)=1/4$ and $V_0^L=\left]0,\frac{1}{2}\right[$. Suppose $f_L(m)$ and V_m^L are defined for $m<n$ such that for all distinct $m',m<n$

- $f_L(m)$ is the middle point of V_m^L,
- $m'<_L m \iff f_L(m')<f_L(m)$,
- $V_{m'}^L\cap V_m^L=\varnothing$,
- $\sup V_m^L<1$,
- if $m'<_L m$, then $\inf V_m^L-\sup V_{m'}^L=0 \iff (m',m)\in S_L$.

There are two cases: either $m<_L n$ for all $m<n$ or there are $m',m<n$ such that $m'<_L n<_L m$ and there is no $k<n$ which is $<_L$-between m' and m. In the first case let m_0 be the $<_L$-largest among all $m<n$. Let $f_L(n)=\sup V_{m_0}^L+\frac{1}{3}(1-\sup V_{m_0}^L)$ and if $(m_0,n)\in S$, then let $V_n^L=B(f_L(n),f_L(n)-\sup V_{m_0}^L)$ and otherwise let $V_n^L=B(f_L(n),\frac{1}{2}(f_L(n)-\sup V_{m_0}^L))$. In the second case define

$$f_L(n)=\begin{cases}\frac{1}{2}(\sup V_{m'}^L+\inf V_m^L) & \text{if } (m',n)\in S \iff (n,m)\in S_L\\ \sup V_{m'}^L+\frac{1}{3}(\inf V_m^L-\sup V_{m'}^L) & \text{if } (m',n)\in S_L \text{ and } (n,m)\notin S_L\\ \sup V_{m'}^L+\frac{2}{3}(\inf V_m^L-\sup V_{m'}^L) & \text{if } (m',n)\notin S_L \text{ and } (n,m)\in S_L.\end{cases}$$

Then define

$$V_n^L=\begin{cases}B(f_L(n),\frac{1}{2}(\inf V_m^L-\sup V_{m'}^L)) & \text{if } (m',n)\in S_L \text{ and } (n,m)\in S_L\\ B(f_L(n),\frac{1}{4}(\inf V_m^L-\sup V_{m'}^L)) & \text{if } (m',n)\notin S_L \text{ and } (n,m)\notin S_L\\ B(f_L(n),f_L(n)-\sup V_{m'}^L) & \text{if } (m',n)\in S_L \text{ and } (n,m)\notin S_L\\ B(f_L(n),\inf V_m^L-f_L(n)) & \text{if } (m',n)\notin S_L \text{ and } (n,m)\in S_L.\end{cases}$$

This ensures the following:

1. for every distinct $n,m\in\mathbb{N}$ the sets V_n^L,V_m^L are disjoint and $f_L(n)$ is the middle point of V_n^L,
2. $\bigcup_{n\in\mathbb{N}}V_n^L$ is dense in $[0,1]$,
3. if $L\restriction\{0,\dots,n\}=L'\restriction\{0,\dots,n\}$, then $f^L\restriction\{0,\dots,n\}=f^{L'}\restriction\{0,\dots,n\}$ and $V_m^L=V_m^{L'}$ for all $m\leqslant n$.
4. $\sup V_n^L=\inf V_m^L \iff n<_L m$ and $(n,m)\in S_L$

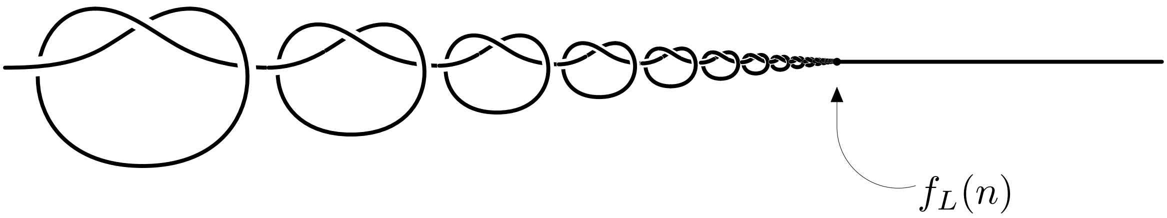

Figure 1: The singular arc whose copies are concatenated to form the knot $F(L)$. Figure produced by MetaPost.

5. $f_L(m') < f_L(m) \iff m' <_L m$.

Now pick open intervals $U_n^L \subset V_n^L$ such that $\overline{U_n^L} \subset V_n^L$ and $f_L(n)$ is the center of U_n^L. Thinking of $\mathbb{R}$ as the x-axis of $\mathbb{R}^3$, replace each U_n^L by the open ball B_n^L with the center at $f_L(n)$ and radius $\frac{1}{2}|U_n^L|$. Now build the knot $K = F(L) \subset S^3$ as follows. Let $K_a^L = (\mathbb{R} \setminus [0,1]) \cup \{\infty\}$. For each $n \in \mathbb{N}$ let $K_n^L \subset \overline{B_n^L}$ be a proper arc in $\overline{B_n^L}$ with end-points coinciding with the end-points of U_n^L. This arc K_n^L is equivalent to the knot depicted on Figure 1, it is a knot sum of infinitely many trefoils and has one singularity precisely at $f_L(n)$. Let K_r^L cover the rest: $K_r^L = [0,1] \setminus \bigcup_{n\in\mathbb{N}} U_n^L$. Now define

$$K = F(L) = \bigcup_{x \in \{a,r\} \cup L} K_x^L.$$

Let us show that F is continuous. Let $L \in \mathrm{LO}^*$. We will show that for every open neighborhood U of $F(L)$ there is an open neighborhood of L which is mapped inside U. Suppose that U is the ε-ball with center $F(L)$ in the Hausdorff metric, cf. 1. Pick n so large that $|V_n^L| < \varepsilon/2$. We claim that for any L' with $L' \restriction \{0,\dots,n\} = L \restriction \{0,\dots,n\}$ we have $F(L') \subset B(F(L),\varepsilon)$ and $F(L) \subset B(F(L'),\varepsilon)$. The argument is symmetric, so we will just show the first part. If x is in $F(L')$, then it is in one of the sets $K_z^{L'}$ for $z \in \{a,r\} \cup \mathbb{N}$. If $z \in \{a,0,\dots,n\}$, then by 3, $x \in F(L)$. Suppose $x \in K_m^{L'}$ for some $m > n$. Denote by $\operatorname{pr} x$ the projection of x onto the x-axis, so we have $d(x, \operatorname{pr} x) < \varepsilon/2$. On the other hand $\operatorname{pr} x \notin \bar{B}_k^L$ for any $k \leqslant n$, so $d(\operatorname{pr} x, F(L)) < \varepsilon/2$ and we have $x \in B(F(L),\varepsilon)$.

Suppose $L \cong L'$ are isomorphic structures in LO^* and let $g\colon \mathbb{N} \to \mathbb{N}$ be a witnessing isomorphism. It induces an order preserving map h_0 from $f_L[L]$ onto $f_{L'}[L']$ defined by $f_L(n) \mapsto f_L(g(n))$. Let

$$W_n^L = \overline{V_n^L} \times \mathbb{R}^2$$

and

$$W_n^{L'} = \overline{V_n^{L'}} \times \mathbb{R}^2.$$

Call these sets "walls". Restricted to W_n^L, the domain of h_0 is just the singleton $\{f_L(n)\}$ which is mapped to $\{f_{L'}(g(n))\}$. Let $h_1^n\colon \overline{B_n^L} \to \overline{B_{g(n)}^{L'}}$ be a homeomorphism such that $h_1(f_L(n)) = f_{L'}(g(n)) = h_0(f_L(n))$, which takes K_n^L to $K_{g(n)}^{L'}$. This is possible

because these arcs are equivalent. Let $M_n = \frac{1}{2}\max\{|V_n^L|, |V_{g(n)}^{L'}|\}$ and let

$$h_2^n \colon \overline{V_n^L} \times [-M_n, M_n]^2 \to \overline{V_{g(n)}^{L'}} \times [-M_n, M_n]^2$$

be a homeomorphism which extends h_1^n and with the property that for all $x \in [-M_n, M_n]^2$ and z_1, z_2, z_1', z_2' the left and right endpoints of V_n^L and $V_{g(n)}^{L'}$ respectively we have

$$h_2^n(z_i, x) = (z_i', x).$$

Finally define a homeomorphism $h_3^n \colon W_n^L \to W_{g(n)}^{L'}$ which extends h_2^n such that

$$\begin{array}{l}\text{For every } (x, y_1, y_2) \in V_n^L \times \mathbb{R}^2 \text{ with } \max\{y_1, y_2\} > M_n \\ \text{we have } h_3(x, y_1, y_2) = (x', y_1, y_2) \text{ for some } x'.\end{array} \tag{P}$$

Consider a pair (n, m) such that $(n, m) \in S_L$ and $n <_L m$. Then by the definitions above for all $x \in W_n^L \cap W_m^L$ we will have

$$h_3^n(x) = h_3^m(x) \tag{Q}$$

Let h_a be the identity on $\mathbb{R}^3 \setminus [0,1] \times \mathbb{R}^2$. Consider x which is a cluster point of $\operatorname{Im} f_L$. Let x' be the corresponding point in $\operatorname{Im} f_L$. Define $h_r(x, y_1, y_2) = (x', y_1, y_2)$ for all $(y_1, y_2) \in \mathbb{R}^2$. By (P) and (Q) above and noting that $M_n \overset{k\to\infty}{\longrightarrow} 0$, the union

$$h = h_a \cup h_r \cup \bigcup_{n \in \mathbb{N}} h_3^n$$

is a well defined function and is a homeomorphism.

On the other hand suppose there exists a homeomorphism h which takes $F(L)$ to $F(L')$. Without loss of generality we can assume that $h(\infty) = \infty$, i.e. h restricts to a homeomorphism of $\mathbb{R}^3$. Let Σ_L be $\{x \in F(L) \mid x \text{ is a singularity}\}$, $I\Sigma_L$ be the set of isolated points in Σ_L and similarly define $\Sigma_{L'}$ and $I\Sigma_{L'}$ for L'. A singular point can only go to a singular point in a homeomorphism because all other points are locally unknotted but every neighborhood of a singular point contains trefoils as components (Definition 2.10). Hence $h \restriction \Sigma_L$ is a homeomorphism onto $\Sigma_{L'}$. $I\Sigma_L$ is a subset of the x-axis which we identified with $\mathbb{R}$, so we can define the order $<_\mathbb{R}$ on $I\Sigma_L$ and by 3, $f_n(L)$ is an order isomorphism from $(\mathbb{N}, <_L)$ to $(I\Sigma_L, <_\mathbb{R})$ and same for L'. Being an isolated point is a topological property, so $h \restriction I\Sigma_L$ is in fact a homeomorphism onto $h \restriction I\Sigma_{L'}$. Also, it has to preserve the betweenness relation: suppose x, y, z are points in $I\Sigma_{L'}$ such that $x <_\mathbb{R} y <_\mathbb{R} z$. Now x and z divide the knot into two arcs one of which is bounded in $\mathbb{R}^3$ and y is in this bounded arc. If h mapped y such that it is not $<_\mathbb{R}$-between $h(x)$ and $h(z)$, then it would mean that $h(y)$ is mapped to the unbounded component of $F(L') \setminus \{h(x), h(z)\}$ which is a contradiction. This implies that $(f_{L'})^{-1} \circ (h \restriction I\Sigma_L) \circ f_L$ is a bijection which is either order-preserving or order-reversing (because it preserves betweenness). But since we assumed that the elements of LO^* have a smallest element but no largest, mirror images are out of the question, so it is in fact an isomorphism. $\square$

4 Reducing a Turbulent Relation to E_K

Recall the equivalence relation E^* from Definition 2.7. The proof of the following theorem can be found after all the preparations on page 26.

4.1 Theorem. *The equivalence relation E^* on $(S^1)^{\mathbb{N}}$ (cf. Definition 2.7) is continuously reducible to E_K.*

Let us begin with some preliminaries in topology. For metric spaces X and Y, where Y compact, let $C(X,Y)$ be the space of all continuous maps from X to Y equipped with the sup-metric. A map $f\colon X \to Y$ is *L-Lipschitz*, for some $L > 0$, if for all $x, y \in X$

$$d_Y(f(x), f(y)) \leqslant L d_X(x, y).$$

The function f is *L-colipschitz*, if for all $x, y \in X$

$$d_Y(f(x), f(y)) \geqslant \frac{1}{L} d_X(x, y).$$

The function f is L-bilipschitz, if it is both L-Lipschitz and L-colipschitz. Note that the notion of L-bilipschitz makes sense only when $L \geqslant 1$. Also note that if f is L-colipschitz and onto, then f has an inverse and this inverse is L-Lipschitz. Thus, one can say that an onto function f is L-bilipschitz if both f and its inverse are L-Lipschitz.

A map between metric spaces is *locally L-Lipschitz* if for every point in the domain there exists ε so that the function is L-Lipschitz restricted to the ε-neighborhood of this point. Analogously define *locally L-colipschitz* and *locally L-bilipschitz*. The length of a path in a metric space is defined through the supremum of the lengths of finite polygons approximating the path. Assuming that there are paths of finite length between any two given points (call such spaces *length-spaces*), one can define the *path metric d^p* given by the infimum of all path lengths from one point to another. All spaces in this paper are length spaces. The following is a standard theorem of metric topology.

4.2 Proposition. *Suppose $f\colon X \to Y$ is a locally L-Lipschitz homeomorphism for length spaces X, Y. Then f is L-Lipschitz as a function from from (X, d^p_X) to (Y, d^p_Y) where d^p_X and d^p_Y are path metrics on X and Y respectively. Same for "Lipschitz" replaced by "colipschitz".* $\square$

Denote by $D^2_r = \bar{B}_{\mathbb{R}^2}(0, r)$ the two dimensional closed disk with radius r. Let T_r be the solid torus $T_r = D^2_r \times S^1$. The metric on this torus is the product metric: $d_{T_r}((x,s),(y,t)) = \sqrt{d_{\mathbb{R}^2}(x,y)^2 + d_{S^1}(x,y)^2}$. Note that from our parametrization of S^1 (Definition 2.7) this metric coincides with the path metric on T_r, i.e. it is in a sense "convex".

We define a tubular neighborhood in a little bit non-standard way. For a proper arc $f\colon I \to \bar{B}_0$, as in Definition 2.10, the set $T(f, \varepsilon) = \bar{B}_{S^3}(f[I], \varepsilon) \cap \bar{B}_0$ is a *tubular neighborhood* of f if there exists an embedding $h\colon D^2_\varepsilon \times I \to S^3$ such that

- $\operatorname{Im} h \cap \bar{B}_0 = T(f, \varepsilon)$,

- $h(\bar{0}, s) = f(s)$ for all $s \in I$,
- for all $\varepsilon' < \varepsilon$ and $x \in \mathrm{Im}(f)$ the set $B(x, \varepsilon') \cap f[I]$ is connected and represents an unknotted proper arc in $B(x, \varepsilon)$,
- $h \restriction D_r^2 \times \{s\}$ is an isometry onto the disc $D \subset \mathbb{R}^3$ which is orthogonal to f at $f(s)$ and whose middle point is $f(s)$.

Similarly define this for a knot $f\colon S^1 \to \mathbb{R}^3$ just replacing I with S^1. The following is a standard fact in knot theory and differential topology in general:

4.3 Theorem. *For every smooth proper arc or a knot f there is $\varepsilon > 0$ such that $T(f, \varepsilon)$ is a tubular neighborhood of f.*

Proof. Let us prove this for a knot $f\colon S^1 \to \mathbb{R}^3$, so the result for a proper arc follows: every proper arc can be extended to a knot by connecting the end-points in the ambient space. We can assume without loss of generality that $|f'(s)| > 0$ for all s. Let $G \subset S^2$ be the set of all the directions of the gradient:

$$G = \Big\{\frac{f'(s)}{|f'(s)|} \mid s \in S^1\Big\}.$$

Since f is smooth and S^1 compact, G is nowhere dense, so there exists $s_0 \in S^2 \setminus G$. For every $s \in S^1$ let M_s be the 2-dimensional subspace of $\mathbb{R}^3$ orthogonal to $f'(s)$ and let n_s be the orthogonal projection of s_0 to M_s which is normalized to length 1 (since $s_0 \notin G$, the orthogonal projection is non-zero). Since f is smooth, $s \mapsto n_s$ is also smooth and n_s is normal to $f'(s)$. Let

$$b_s = \frac{f'(s) \times n_s}{|f'(s) \times n_s|}.$$

Now define the following map:

$$g\colon \mathbb{R}^2 \times S^1 \to \mathbb{R}^3,$$

$$g(x, y, s) = x n_s + y b_s + f(s).$$

By the smoothness of f, $s \mapsto n_s$ and $s \mapsto b_s$, g is also smooth. By smoothness and compactness one finds ε such that restricted to $D_\varepsilon^2 \times S^1$, g is injective. It is easy to see that all the conditions for tubular neighborhood are satisfied in particular because $f[S^1]$ is now a strong deformation retract of $g[D_\varepsilon^2 \times S^1]$, so restricted to $D_\varepsilon^2 \times S^1$, g witnesses that $T(f, \varepsilon)$ is a tubular neighborhood. $\square$

If ε is such that $T(f, \varepsilon)$ is a tubular neighborhood, we say that ε is *tubular for f*. It is easy to see that if ε is tubular for f, then so is every $\varepsilon' < \varepsilon$.

4.4 Lemma. *Suppose $f\colon I \to \bar{B}$ is a smooth proper arc in a closed ball $\bar{B} = \bar{B}(x, r)$. Suppose ε is tubular for f. Then any tame proper arc $f_0 \in B_{C(I,\bar{B})}(f, \varepsilon)$ has f as a component.*

Proof. Suppose $f_0 \in B_{C(I,\bar{B})}(f,\varepsilon)$ is a tame proper arc. Without loss of generality $I=[0,1]$ is the unit interval. Suppose $g\colon D^2_\varepsilon \times I \to S^3$ is a homeomorphism witnessing that $T(f,\varepsilon)$ is a tubular neighborhood. There is a δ_0 such that

$$(D^2_\varepsilon \times [0,\delta_0]\,,\gamma_0),$$

where $\gamma_0 = \mathrm{Im}(g^{-1}\circ f_0) \cap D^2_\varepsilon \times [0,\delta_0]$, is unknotted. Let δ_1 be so small that

$$(\bar{B}(x,r-\delta_1), f \cap \bar{B}(x,r-\delta_1))$$

is a connected proper arc equivalent to $(\bar{B},f)$. This can be found again by compactness and smoothness. Let δ be the minimum of δ_0 and the following two:

$$\sup\{\delta_2 \mid f[0,\delta_2] \cap \bar{B}(x,r-\delta_1) = \varnothing\}.$$

$$\sup\{\delta_3 \mid f[1-\delta_3,1] \cap \bar{B}(x,r-\delta_1) = \varnothing\}.$$

Let $\varepsilon' < \varepsilon$ be such that $g^{-1}\circ f_0 \subset D^2_{\varepsilon'} \times I$ which exists by compactness and the fact that $g^{-1}\circ f_0$ is in the interior of $D^2_\varepsilon \times I$. Denote $\gamma = \mathrm{Im}(g^{-1}\circ f_0) \cap D^2_{\varepsilon'} \times [0,\delta]$ Since now

$$(D^2_{\varepsilon'} \times [0,\delta]\,,\gamma)$$

is an unknotted proper arc, there is a homeomorphism of pairs

$$H\colon (D^2_{\varepsilon'} \times [0,\delta]\,,\gamma) \to (D^2_{\varepsilon'} \times [0,1-\delta]\,,\{0,0\} \times [0,1-\delta]).$$

Extend H to $H_1\colon D^2_\varepsilon \times I \to D^2_\varepsilon \times I$ so that H_1 fixes $(\partial D^2_\varepsilon)\times I$ point-wise. Now $g \circ H_1 \circ g^{-1}$ can be extended to a homeomorphism of $\bar{B}$ (by just identity outside $\mathrm{Im}\, g$) witnessing that f_0 has f as a component. □

For our purposes we have to redefine the notion of a component of a knot. In Definition 2.10 we gave a fairly standard definition of what that means. However, it is not suitable for the kind of fractal knots we are dealing with in this section. For example the knot $K(\bar{x})$ that we are going to define is not smooth at any point, so it will not contain any tame knot as a component. On the other hand it is built as a limit of tame knots K_n such that every prime knot appears as a component of K_n for sufficiently large n. So we want to redefine the notion of a component to apply to such wild knots in a more natural way.

4.5 Definition. Let $f\colon I \to \bar{B}$ be a proper arc (not necessarily tame) and g another proper arc. We say that f *has g strongly as a component* if there exists an ε such that every tame proper arc $f_1 \in B_{C(I,\bar{B})}(f,\varepsilon)$ has g as a component. By the above Lemma, if f is smooth, then it has g strongly as a component if and only if it has it as a component, and this is witnessed by any ε which is tubular for f.

Having g strongly as a component is an invariant property:

4.6 Lemma. *If $f\colon I \to \bar{B}$ has g strongly as a component and $H\colon \bar{B} \to \bar{B}'$ is a homeomorphism, then also $(\bar{B}', H\circ f)$ has g strongly as a component.*

Proof. Suppose ε witnesses that f has g strongly as a component. H induces a homeomorphism $\bar{H}$ of $C(I,\bar{B})$, so there is ε' such that

$$\bar{H}^{-1}[B_{C(I,\bar{B})}(\bar{H}(f),\varepsilon')] \subset B_{C(I,\bar{B})}(f,\varepsilon).$$

Given any tame arc $h \in B_{C(I,\bar{B})}(H \circ f, \varepsilon')$, it is equivalent to $\bar{H}^{-1}(h) = H^{-1} \circ h$ which is a tame arc in $B_{C(I,\bar{B})}(f,\varepsilon)$ and so by the definition of having g strongly as a component, has g as a component. □

4.7 Lemma. *Let $f\colon I \to \bar{B}$ be a proper arc and suppose that it has K strongly as a component which is witnessed by ε. Suppose $f'\colon I \to \bar{B}$ satisfies $d(f',f) < \varepsilon/2$. Then f' has K strongly as a component.*

Proof. Let g be any arc such that $g \in B_{C(I,\bar{B})}(f',\varepsilon/2)$. Then $d(f',g) < \varepsilon/2$ and by the triangle inequality we have $d(g,f) < \varepsilon$ and so $g \in B_{C(I,\bar{B})}(f,\varepsilon)$. By the definition of having K strongly as a component, we see that g has K as a component, and so $\varepsilon/2$ witnesses that f' has K strongly as a component. □

Before the next Lemma, let us state a well known fact from differential geometry:

4.8 Fact. *(Folklore) The operator norm of an $(n \times n)$-matrix A is given by*

$$\sup\{|A\bar{x}|\colon |\bar{x}| = 1, \bar{x} \in \mathbb{R}^n\}.$$

A smooth function f from n-manifold N to n-manifold M is locally $(L+\varepsilon)$-Lipschitz at $x \in N$ for all $\varepsilon > 0$ if and only if the operator norm of the Jacobian at x is at most L. □

4.9 Lemma. *If $f\colon S^1 \to \mathbb{R}^3$ is a smooth curve which is locally L-bilipschitz for some $L \geqslant 1$, then for every $\varepsilon > 0$ it has a tubular neighborhood which is realized by a locally $(L+\varepsilon)$-bilipschitz homeomorphism $g\colon D^2_r \times S^1 \to T(f,r)$ for some r.*

Proof. Let g be as in the proof of Theorem 4.3. Denote $n_s = (n_s^1, n_s^2, n_s^3)$ and $b_s = (b_s^1, b_s^2, b_s^3)$ for all s and $f(s) = (f_1(s), f_2(s), f_3(s))$. The Jacobian of g at $(0,0,s)$ is then

$$J(s) = \begin{pmatrix} n_s^1 & b_s^1 & f_1'(s) \\ n_s^2 & b_s^2 & f_2'(s) \\ n_s^3 & b_s^3 & f_3'(s) \end{pmatrix}.$$

Let $\bar{x} \in \mathbb{R}^3$ be any unit vector, $|\bar{x}| = 1$, $\bar{x} = (x_1, x_2, x_3)$. Then

$$J(s)\bar{x} = x_1 n_s + x_2 b_s + x_3 f'(s).$$

Since $L \geqslant 1$, $|f'(s)| \geqslant 1$, $|n_s| = |b_s| = 1$ and all these vectors are orthogonal to each other, we have:

$$\begin{aligned} |J(s)\bar{x}| &= |x_1 n_s + x_2 b_s + x_3 f'(s)| \\ &= \sqrt{x_1^2 + x_2^2 + x_3^2 |f'(s)|^2} \\ &\leqslant \sqrt{x_1^2|f'(s)|^2 + x_2^2|f'(s)|^2 + x_3^2|f'(s)|^2} \\ &= |f'(s)|\sqrt{x_1^2 + x_2^2 + x_3^2} \\ &= |f'(s)|. \end{aligned}$$

This means that the vector $\bar{x} = (0,0,1)$ maximizes the norm of $J(s)\bar{x}$ whence it is $|f'(s)|$ and so this means that the operator norm of $J(s)$ is $|f'(s)|$. By the continuity of the Jacobian and compactness, we can find r such that the Jacobian of g at (x,y,s) for all $(x,y) \in D_r^2$ has operator norm at most $|f'(s)|+\varepsilon/2$. Then restricted to $D_r^2 \times S^1$, g will be locally $(|f'(s)|+\varepsilon)$-Lipschitz. Consider then the inverse $J(s)^{-1}$. Let $\bar{x}$ be arbitrary unit vector in the coordinate system $(n_s, b_s, f'(s)/|f'(s)|)$ and consider

$$\begin{aligned} |J(s)^{-1}(x_1 n_s + x_2 b_s + x_3 f'(s)/|f'(s)|)| &= |J(s)^{-1}J(s)(x_1,x_2,x_3/|f'(s)|)| \\ &= |(x_1,x_2,x_3/|f'(s)|)| \\ &\leqslant |(x_1,x_2,x_3)| \\ &= 1 \end{aligned}$$

where the last inequality follows from the assumption that $|f'(s)| \geqslant 1$. So the operator norm of $J(s)^{-1}$ is 1. By moving to yet smaller r, one can ensure that the operator norm of g^{-1} is everywhere at most $1+\varepsilon/2 \leqslant L+\varepsilon/2$. Thus, for this r, both g and g^{-1} are locally $(L+\varepsilon)$-Lipschitz, and so g is locally $(L+\varepsilon)$-bilipschitz. □

For a sequence of functions (f_n) such that $\operatorname{Im} f_n \subset \operatorname{dom} f_{n+1}$ for all n, denote by $\bigcirc_{k=i}^{j} f_k = f_i \circ \cdots \circ f_j$ their composition. The order matters in composition, so if $j < i$, this notation simply means that the functions are taken in reverse order, for example one could write

$$\left(\bigcirc_{i=2}^{5} f_i\right)^{-1} = \bigcirc_{i=5}^{2} f_i^{-1}.$$

Otherwise we often omit the circle from the composition notation, writing just $f \circ g = fg$. We also sometimes omit the brackets from the function notation, i.e. $f(x) = fx$.

Let $(K_n)_{n\in\mathbb{N}}$ enumerate all (tame) prime knot types (trivial knot not included). The content of the following proposition is partially illustrated in Figure 2.

4.10 Proposition. *There exist sequences (ε_n), (λ_n), (L_n), (g_n) and (f_n) such that for all n,*

(a) $g_n \colon T_{\varepsilon_{n+1}} \to T_{\varepsilon_n}$, $f_n \colon S^1 \to T_{\varepsilon_n}$ and $g_n(\bar{0}, s) = f_n(s)$ for all $s \in S^1$,

(b) $\varepsilon_0 = 1$, $L_n < 2^{2^{-n}}$ and $4\lambda_{n+1} < 2\varepsilon_{n+1} < \lambda_n < \varepsilon_n \leqslant 2^{-n}$,

(c) $20\varepsilon_{n+1}$ is tubular for f_n,

(d) $\operatorname{Im} g_n = T(f_n, \varepsilon_{n+1}) \subset T_{\lambda_n}$ and g_n witnesses that $T(f_n, \varepsilon_{n+1})$ is a tubular neighborhood of f_n,

(e) $f_n(s) = s$ for $s \notin B_{S^1}(0, \lambda_n)$ and $f_n(s) \in B_{T_{\varepsilon_n}}(\bar{0}, \lambda_n)$ for $s \in B_{S^1}(0, \lambda_n)$,

(f) $(B_{T_{\varepsilon_n}}(\bar{0}, \lambda_n), f_n \cap B_{T_{\varepsilon_n}}(\bar{0}, \lambda_n))$ is a proper arc of knot type K_n,

(g) f_n is locally 1-colipschitz,

(h) g_n is locally L_n-bilipschitz,

(i) g_n *is* $\varepsilon_n^2/\varepsilon_{n+1}$*-colipschitz.*

Proof. Let $\varepsilon_0 = 1$, $\lambda_0 = 1/2$, $L_0 = 3/2$ and f_0 be a smooth map $S^1 \to T_{\varepsilon_0}$ satisfying (e) and (f). By making the knot sufficiently small, we can make sure that the length of the curve is arbitrarily close to the length of S^1, so that f_0 is locally L_0'-bilipschitz for some $L_0' < L_0$ and we can also make sure that the gradient is always at least 1, i.e. f_0 is 1-colipschitz, so (g) is satisfied. As an induction hypothesis assume that $\varepsilon_m, \lambda_m, L_m, f_m$ are defined for all $0 \leqslant m \leqslant n$ and g_m is defined for all $0 \leqslant m < n$. If $m < n$, then all conditions are satisfied. For $\varepsilon_n, \lambda_n, L_n, f_n$ only (e), (f) and (g) are satisfied and additionally f_n is locally L_n'-bilipschitz for some $L_n' < L_n$. Let ε_{n+1}' be so small that $20\varepsilon_{n+1}'$ is tubular for f_n, that $\bar{B}(f_n, \varepsilon_{n+1}') \subset T_{\lambda_n}$, that a tubular neighborhood of thickness ε_{n+1}' can be realized by a locally L_n-bilipschitz function $g_n' \colon T_{\varepsilon_{n+1}'} \to T_{\varepsilon_n}$ and so that $2\varepsilon_{n+1}' < \min\{\lambda_n, 2^{-n-2}\}$. These are possible respectively by Theorem 4.3, by compactness, by Lemma 4.9 and the rest trivially. So let g_n' be this function. Now g_n' satisfies (d) and (h), so everything except (i) is satisfied. By compactness and the fact that it is locally L_n-colipschitz, there exists ε so that g_n' is ε^{-1}-colipschitz. Let $\varepsilon_{n+1} \leqslant \varepsilon_{n+1}'$ be so small that $\varepsilon_{n+1}/\varepsilon_n^2 < \varepsilon$. Then let $g_n = (g_n' \restriction T_{\varepsilon_{n+1}})$. Now $f_n = g_n \restriction S^1$ and all conditions are satisfied. Now define $\lambda_{n+1} < \varepsilon_{n+1}/2$, L_{n+1} anything between 1 and $2^{2^{-n-1}}$ and f_{n+1} to be from S^1 to $T_{\varepsilon_{n+1}}$ satisfying (e), (f) and (g) and which is locally L_{n+1}'-colipschitz for some $L_{n+1}' < L_{n+1}$. □

Let $\theta^s \colon T_1 \to T_1$ be the rotation by s: $\theta^s(b,t) = (b, t+s)$. Note that $\theta^s\theta^t = \theta^{s+t}$. Let H_1^s be the homeomorphism of T_1 defined by: $H_1^s(b,t) = \theta^{r(b,s)}(b,t)$ where $r(b,s) = (1 - 2\max\{0, |b| - 1/2\})s$ and let $H_\varepsilon^s = \beta_\varepsilon H^s \beta_\varepsilon^{-1}$ where $\beta_\varepsilon \colon T_1 \to T_\varepsilon$ is the homeomorphism $\beta_\varepsilon(b,s) = (\varepsilon b, s)$. It is clear that

$$H_\varepsilon^s \restriction T_{\varepsilon/2} = \theta^s \tag{2}$$

and for all $x \in T_{\varepsilon/2}$ we have

$$d(x, H_\varepsilon^s(x)) = d(x, \theta^s(x)) = |s| \tag{3}$$

and in general $d(x, H_\varepsilon^s(x)) \leqslant |s|$. Also for all $x, y \in T_\varepsilon$ we have

$$d(H_\varepsilon^s(x), H_\varepsilon^s(y)) \leqslant d(x,y) + |s|. \tag{4}$$

Let $\hat{L}_n = \prod_{i=0}^n L_n$. By (b),

$$\hat{L}_n < 5. \tag{5}$$

Let $\bar{x}$ be a sequence in $(S^1)^{\mathbb{N}}$. Define $\hat{g}_n^{\bar{x}} \colon T_{\varepsilon_{n+1}} \to T_1$ by

$$\hat{g}_n^{\bar{x}} = \bigcirc_{k=0}^{n} \theta^{x_k} g_k \theta^{-x_k}. \tag{6}$$

It is easy to calculate that $\hat{g}_n^{\bar{x}}$ is locally $\hat{L}_n$-bilipschitz, because θ^s is an isometry and by (h). By (5) it is locally 5-bilipschitz. In particular we have:

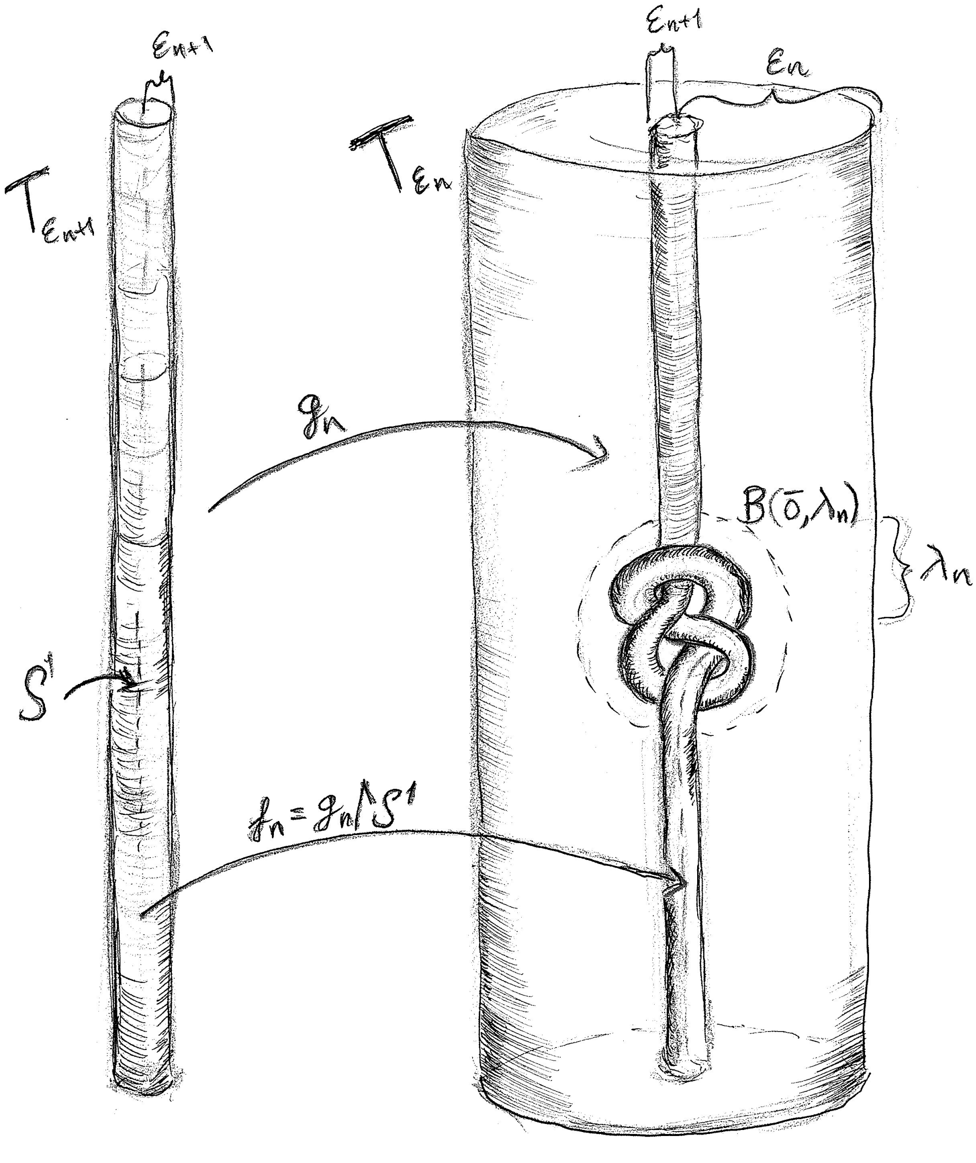

Figure 2: An illustration of Proposition 4.10.

4.11 Proposition. *$\hat{g}_n^{\bar{x}}$ is 5-Lipschitz.*

Proof. This follows from the fact that the standard metric on $\operatorname{dom} g_n^{\bar{x}} = T_{\varepsilon_{n+1}}$ coincides with its path metric and Proposition 4.2. □

Equip each $\operatorname{Im}(\hat{g}_n)$ with the path metric $d^{p(n)}$. From Proposition 4.2 we obtain:

4.12 Proposition. *$\hat{g}_n$ is 5-bilipschitz as a function from $T_{\varepsilon_{n+1}}$ to $(\operatorname{Im}\hat{g}_n, d^{p(n)})$.* □

Let $\hat{f}_n^{\bar{x}} = \hat{g}_n^{\bar{x}} \restriction S^1$ and define

$$K(\bar{x}) = \bigcap_{n \in \mathbb{N}} \operatorname{Im} \hat{g}_n^{\bar{x}} \tag{7}$$

A slight modification of this is going to be the knot associated with $\bar{x}$ in the reduction. First let us show that $K(\bar{x})$ is indeed a knot and is in fact the image of $f^{\bar{x}}$ which is the limit of the sequence $(\hat{f}_n^{\bar{x}})$.

4.13 Proposition. *$K(\bar{x})$ is a knot, i.e. an injective continuous image of S^1.*

Proof. For the time of this proof let us drop the "$\bar{x}$" from the upper case: $\hat{g}_n^{\bar{x}} = \hat{g}_n$, $\hat{f}_n^{\bar{x}} = \hat{f}_n$.

$$\begin{aligned}
d(\hat{f}_n(s), \hat{f}_{n+1}(s)) &= d(\hat{g}_n(s), \hat{g}_n \theta^{x_{n+1}} g_{n+1} \theta^{-x_{n+1}}(s)) \\
&\leqslant 5d(s, \theta^{x_{n+1}} f_{n+1} \theta^{-x_{n+1}}(s)), && (8) \\
&= 5d(\theta^{-x_{n+1}} s, f_{n+1} \theta^{-x_{n+1}}(s)), && (9) \\
&= 5d(t, f_{n+1}(t)), \\
&\leqslant 10\lambda_{n+1} && (10)
\end{aligned}$$

where $t = \theta^{-x_{n+1}}(s)$. (8) follows from Proposition 4.11, (9) follows from the fact that θ^s is an isometry and (10) follows from (e) and (f). From (b) it follows in particular that $\sum_{k=n}^{\infty} \lambda_k \leqslant \varepsilon_n$, so from the arbitrariness of n above, the sequence $(\hat{f}_n)$ is Cauchy. The limit f is contained in $\operatorname{Im}\hat{g}_n$ for every n, so $\operatorname{Im} f \subset K(\bar{x})$ on one hand, and if a point is in $\operatorname{Im}\hat{g}_n$, then its distance from $\operatorname{Im} f$ is at most ε_n which tends to zero, so $K(\bar{x}) \subset \operatorname{Im} f$ on the other.

It follows from (i) that the limit, f, is injective: Let $s, t \in S^1$ and let $\varepsilon = d(s,t)$. Let n be so large that $20\varepsilon_{n-1} < \varepsilon$. From the above we know that $d(\hat{f}_n, f) \leqslant 10 \sum_{m>n} \lambda_n$ which is at most $10\varepsilon_{n+1}$ by (b). Now

$$\begin{aligned}
d(\hat{f}_n(s), \hat{f}_n(t)) &\leqslant d(\hat{f}_n(s), f(s)) + d(f(s), f(t)) + d(\hat{f}_n(t), f(t)) \\
&\leqslant 20\varepsilon_{n+1} + d(f(s), f(t)).
\end{aligned}$$

So we have

$$d(f(s), f(t)) \geqslant d(\hat{f}_n(s), \hat{f}_n(t)) - 20\varepsilon_{n+1}.$$

But by (i), $\hat{f}_n$ is M^{-1}-colipschitz, where

$$M = \prod_{m=1}^{n} \frac{\varepsilon_m}{\varepsilon_{m-1}^2} \geqslant \varepsilon_n / \varepsilon_{n-1}. \tag{11}$$

This is obtained by multiplying the colipschitz constants of the factors of $\hat{f}_n$. Thus

$$\begin{aligned}
d(f(s), f(t)) &\geqslant d(\hat{f}_n(s), \hat{f}_n(t)) - 20\varepsilon_{n+1} \\
&\geqslant \frac{\varepsilon_n}{\varepsilon_{n-1}} d(s,t) - 20\varepsilon_{n+1} \\
&= \frac{\varepsilon_n}{\varepsilon_{n-1}} \varepsilon - 20\varepsilon_{n+1} \\
&\geqslant \frac{\varepsilon_n}{\varepsilon_{n-1}} \cdot 20\varepsilon_{n-1} - 20\varepsilon_{n+1} \\
&\geqslant 20(\varepsilon_n - \varepsilon_{n+1}) \\
&> 0.
\end{aligned}$$

□

Let $\bar{x}'$ be another sequence. Let us again simplify the notation $\hat{g}_n^{\bar{x}} = \hat{g}_n$ and $\hat{g}_n^{\bar{x}'} = \hat{g}'_n$.

For each $n \in \mathbb{N}$ let $y_{n+1} = (x'_{n+1} - x_{n+1}) - (x'_n - x_n)$. Define homeomorphisms of T_1 as follows: Let $H_0 = H_{\varepsilon_0}^{x'_0 - x_0}$ and

$$H_{n+1}(x) = \begin{cases} H_n \hat{g}_n H_{\varepsilon_{n+1}}^{y_{n+1}} \hat{g}_n^{-1}(x) & \text{if } x \in \operatorname{Im}(\hat{g}_n) \\ H_n & \text{otherwise.} \end{cases} \tag{12}$$

This is a homeomorphism for all n which can be seen by induction. Suppose H_n is a homeomorphism, then H_{n+1} is defined in two parts. Suppose x is on the boundary: $x \in \partial \operatorname{Im}(\hat{g}_n)$. But then $\hat{g}_n^{-1} x \in \partial T_{\varepsilon_{n+1}}$ and on this set $H_{\varepsilon_{n+1}}^{y_{n+1}}$ is the identity. Thus $H_n \hat{g}_n H_{\varepsilon_{n+1}}^{y_{n+1}} \hat{g}_n^{-1}(x) = H_n(x)$.

Now if x happens to be in $\operatorname{Im}(\hat{g}_{n+1})$, i.e. $x = \hat{g}_{n+1}(y)$ for some y, then

$$\begin{aligned}
\hat{g}_n^{-1}(x) &= \hat{g}_n^{-1} \hat{g}_{n+1}(y) \\
&= \hat{g}_n^{-1} \hat{g}_n \theta^{x_{n+1}} g_{n+1} \theta^{-x_{n+1}}(y) \\
&= \theta^{x_{n+1}} g_{n+1} \theta^{-x_{n+1}}(y)
\end{aligned}$$

and so $\hat{g}_n^{-1}(x)$ is in $T_{\varepsilon_{n+1}/2}$ by (d) and (b). Therefore by (2), if $x \in \operatorname{Im} \hat{g}_{n+1}$, then we have

$$H_{n+1}(x) = H_n \hat{g}_n \theta^{y_{n+1}} \hat{g}_n^{-1}(x) \tag{13}$$

4.14 Proposition. $H_n \hat{g}_n = \hat{g}'_n \theta^{x'_n - x_n}$.

Proof. Induction:

$$\begin{aligned}
H_0 \hat{g}_0 &= H_{\varepsilon_0}^{x'_0 - x_0} \theta^{x_0} g_0 \theta^{-x_0} \\
&= \theta^{x'_0 - x_0} \theta^{x_0} g_0 \theta^{-x_0} \\
&= \theta^{x'_0} g_0 \theta^{-x'_0} \theta^{x'_0 - x_0} \\
&= \hat{g}'_0 \theta^{x'_0 - x_0}.
\end{aligned} \tag{14}$$

Step (14) follows from the fact that restricted to the range of g_0, $H_{\varepsilon_0}^s = \theta^s$.

For $n+1$:

$$\begin{aligned} H_{n+1}\hat{g}_{n+1} &= H_n\hat{g}_n H_{\varepsilon_{n+1}}^{y_{n+1}}(\hat{g}_n)^{-1}\hat{g}_{n+1} \\ &= H_n\hat{g}_n\theta^{y_{n+1}}(\hat{g}_n)^{-1}\hat{g}_{n+1} \\ &= H_n\hat{g}_n\theta^{y_{n+1}}\theta^{x_{n+1}}g_{n+1}\theta^{-x_{n+1}} \\ &= \hat{g}'_n\theta^{x'_n-x_n}\theta^{y_{n+1}}\theta^{x_{n+1}}g_{n+1}\theta^{-x_{n+1}} \\ &= \hat{g}'_n\theta^{x'_{n+1}}g_{n+1}\theta^{-x'_{n+1}}\theta^{x'_{n+1}-x_{n+1}} \\ &= \hat{g}'_{n+1}\theta^{x'_{n+1}-x_{n+1}}. \end{aligned}$$

□

In particular, since θ^s is a bijection, $\mathrm{Im}(H_n\hat{g}_n) = \mathrm{Im}(\hat{g}'_n)$, so $(H_n \restriction \mathrm{Im}\, g_n)$ is a homeomorphism from $\mathrm{Im}(g_n)$ to $\mathrm{Im}(g'_n)$. Since the domain of $\hat{g}_{n+1}$ is $T_{\varepsilon_{n+2}}$ and restricted to this domain we have by (b) and (2)

$$\theta^{x'_{n+1}-x_{n+1}} \restriction T_{\varepsilon_{n+2}} = H_{\varepsilon_{n+1}}^{x'_{n+1}-x_{n+1}} \restriction T_{\varepsilon_{n+2}},$$

we can also write:

$$H_n\hat{g}_n = \hat{g}'_n H_{\varepsilon_{n+1}}^{x'_n-x_n}. \tag{15}$$

Now, when $x \in \mathrm{Im}\,\hat{g}_{n+1}$ we can replace (12) by the equivalent

$$H_{n+1} = \hat{g}'_n\theta^{x'_{n+1}-x_{n+1}}\hat{g}_n^{-1}(x),$$

because

$$\begin{aligned} H_{n+1} &= H_n\hat{g}_n H_{\varepsilon_{n+1}}^{y_{n+1}}\hat{g}_n^{-1} \\ &= H_n\hat{g}_n\theta^{y_{n+1}}\hat{g}_n^{-1} \\ &= \hat{g}'_n\theta^{x'_n-x_n}\theta^{y_{n+1}}\hat{g}_n^{-1} \\ &= \hat{g}'_n\theta^{x'_{n+1}-x_{n+1}}\hat{g}_n^{-1}(x). \end{aligned} \tag{16}$$

Where (16) follows from (13). On the other hand, a similar calculation using (15) gives for $x \in \mathrm{Im}(\hat{g}_n)$

$$H_{n+1} = \hat{g}'_n H_{\varepsilon_{n+1}}^{x'_{n+1}-x_{n+1}}\hat{g}_n^{-1}(x),$$

because

$$\begin{aligned} H_{n+1} &= H_n\hat{g}_n H_{\varepsilon_{n+1}}^{y_{n+1}}\hat{g}_n^{-1} \\ &= \hat{g}'_n H_{\varepsilon_{n+1}}^{x'_n-x_n} H_{\varepsilon_{n+1}}^{y_{n+1}}\hat{g}_n^{-1} \\ &= \hat{g}'_n H_{\varepsilon_{n+1}}^{x'_{n+1}-x_{n+1}}\hat{g}_n^{-1}(x). \end{aligned}$$

So we have

$$H_{n+1}(x) = \hat{g}'_n\theta^{x'_{n+1}-x_{n+1}}\hat{g}_n^{-1}(x), \qquad \text{if } x \in \mathrm{Im}(\hat{g}_{n+1}) \tag{17}$$

$$H_{n+1}(x) = \hat{g}'_n H_{\varepsilon_{n+1}}^{x'_{n+1}-x_{n+1}}\hat{g}_n^{-1}(x), \qquad \text{if } x \in \mathrm{Im}(\hat{g}_n) \tag{18}$$

4.15 Proposition. *Suppose $x, z \in \operatorname{Im} \hat{g}_{n-1}$. Then*

$$d(H_n(x), H_n(z)) \leqslant 25 d^{p(n-1)}(x, z) + 5|x'_n - x_n|.$$

Proof. Using Propositions 4.11 and 4.12, and (18) and (4) we can calculate:

$$\begin{aligned} d(H_n(x), H_n(z)) &= d(\hat{g}'_{n-1} H_{\varepsilon_n}^{x'_n - x_n} \hat{g}_{n-1}^{-1}(x), \hat{g}'_{n-1} H_{\varepsilon_n}^{x'_n - x_n} \hat{g}_{n-1}^{-1}(z)) \\ &\leqslant 5 d(H_{\varepsilon_n}^{x'_n - x_n} \hat{g}_{n-1}^{-1}(x), H_{\varepsilon_n}^{x'_n - x_n} \hat{g}_{n-1}^{-1}(z)) \\ &\leqslant 5 d(\hat{g}_{n-1}^{-1}(x), \hat{g}_{n-1}^{-1}(z)) + 5|x'_n - x_m| \\ &\leqslant 25 d^{p(n-1)}(x, z) + 5|x'_n - x_m|. \end{aligned}$$

□

4.16 Proposition. *Suppose $Y \subset X$ are metric spaces, $\theta \colon X \to X$ is an isometric homeomorphism, and $g \colon Y \to X$ is an embedding. Additionally assume that $\lambda = \sup_{y \in Y} d(y, gy) < \infty$ and $x \in \operatorname{Im} g$, $y \in Y$. Then*

$$d(x, \theta g y) \leqslant d(g^{-1}x, \theta y) + 2\lambda.$$

Proof.

$$\begin{aligned} d(x, \theta g y) &= d(\theta^{-1}x, gy) \\ &\leqslant d(\theta^{-1}x, y) + d(y, gy) \\ &= d(x, \theta y) + d(y, gy) \\ &\leqslant d(g^{-1}x, \theta y) + d(g^{-1}x, x) + d(y, gy) \\ &\leqslant d(g^{-1}x, \theta y) + 2\lambda. \end{aligned}$$

□

4.17 Proposition. *Assume $k < n$ and suppose θ_i and θ'_i are isometries of T_1 of the form θ^s for $i \in \{k, k+1, \dots, n\}$. and $t \in \operatorname{Im} g_n$. Then*

$$d\big(t, \bigcirc_{i=k}^{n} g_i \theta_i \circ \bigcirc_{i=n}^{k} \theta'_i g_i^{-1}(t)\big) \leqslant d\big(t, \bigcirc_{i=k}^{n} \theta_i \theta'_i(t)\big) + 4 \sum_{i=k}^{n} \lambda_i.$$

Proof. Applying Proposition 4.16 $2(n-k+1)$ times in a row one obtains the result. The first of these applications is the special case where $\theta = \theta^0 = \mathrm{id}$:

$$d(x, gy) \leqslant d(g^{-1}x, y) + 2\lambda.$$

□

4.18 Proposition. *Fix n and k and let $x \in \operatorname{Im} \hat{g}_{n+k}$. Then*

$$d(H_n(x), H_{n+k}(x)) \leqslant 10 \sup_{m \geqslant n} |x'_m - x_m| + 20\varepsilon_n.$$

Proof. Since $x \in \operatorname{Im} \hat{g}_{n+k}$, we can write by (17):

$$\begin{aligned} H_n(x) &= \hat{g}'_{n-1} \theta^{x'_n} \theta^{-x_n} \hat{g}_{n-1}^{-1}(x) \\ \text{and } H_{n+k}(x) &= \hat{g}'_{n+k-1} \theta^{x'_{n+k}} \theta^{-x_{n+k}} \hat{g}_{n+k-1}^{-1}(x). \end{aligned}$$

Applying (6) we can further rewrite the expression for H_{n+k}:

$$H_{n+k}(x) = \hat{g}'_{n-1} \bigcirc_{i=n}^{n+k-1} \theta^{x'_i} g_i \theta^{-x'_i} \circ \theta^{x'_{n+k}} \theta^{-x_{n+k}} \hat{g}^{-1}_{n+k-1}(x).$$

Adopting notations

$$\begin{aligned} a &= \theta^{x'_n} \theta^{-x_n} \hat{g}^{-1}_{n-1}(x) \text{ and} \\ b &= \bigcirc_{i=n}^{n+k-1} \theta^{x'_i} g_i \theta^{-x'_i} \circ \theta^{x'_{n+k}} \theta^{-x_{n+k}} \hat{g}^{-1}_{n+k-1}(x) \end{aligned}$$

and using Proposition 4.11 we get:

$$d(H_n(x), H_{n+k}(x)) \leqslant d(\hat{g}'_{n-1}(a), \hat{g}'_{n-1}(b)) \leqslant 5d(a, b). \tag{19}$$

Now we can further rewrite b by expanding the term $\hat{g}^{-1}_{n+k-1}$ using (6) as

$$\begin{aligned} \hat{g}^{-1}_{n+k-1} &= \bigcirc_{i=n+k-1}^{0} \theta^{x_i} g_i^{-1} \theta^{-x_i} \\ &= \bigcirc_{i=n+k-1}^{n} \theta^{x_i} g_i^{-1} \theta^{-x_i} \circ \bigcirc_{i=n-1}^{0} \theta^{x_i} g_i^{-1} \theta^{-x_i} \\ &= \bigcirc_{i=n+k-1}^{n} \theta^{x_i} g_i^{-1} \theta^{-x_i} \circ \hat{g}^{-1}_{n-1} \\ &= \bigcirc_{i=n+k-1}^{n+1} \theta^{x_i} g_i^{-1} \theta^{-x_i} \circ \theta^{x_n} g_n^{-1} \theta^{-x_n} \hat{g}^{-1}_{n-1} \end{aligned}$$

so

$$b = \bigcirc_{i=n}^{n+k-1} \theta^{x'_i} g_i \theta^{-x'_i} \circ \theta^{x'_{n+k}} \theta^{-x_{n+k}} \circ \bigcirc_{i=n+k-1}^{n+1} \theta^{x_i} g_i^{-1} \theta^{-x_i} \circ \theta^{x_n} g_n^{-1} \circ \theta^{-x_n} \hat{g}^{-1}_{n-1}(x).$$

Now, using the substitution $t = \theta^{-x_n} \hat{g}^{-1}_{n-1}(x)$ rewrite both a and b again:

$$\begin{aligned} a &= \theta^{x'_n}(t), \\ b &= \bigcirc_{i=n}^{n+k-1} \theta^{x'_i} g_i \theta^{-x_i} \circ \theta^{x'_{n+k}} \theta^{-x_{n+k}} \bigcirc_{i=n+k-1}^{n+1} \theta^{x_i} g_i^{-1} \theta^{-x_i} \circ \theta^{x_n} g_n^{-1}(t) \end{aligned}$$

Because $\theta^{-x'_n}$ is an isometry, we have

$$d(a, b) = d(\theta^{-x'_n} a, \theta^{-x'_n} b) = d(t, \theta^{-x'_n} b). \tag{20}$$

So let us define $c = \theta^{-x'_n} b$. Now

$$\begin{aligned} c &= \theta^{-x'_n} \bigcirc_{i=n}^{n+k-1} \theta^{x'_i} g_i \theta^{-x'_i} \circ \theta^{x'_{n+k}} \theta^{-x_{n+k}} \bigcirc_{i=n+k-1}^{n+1} \theta^{x_i} g_i^{-1} \theta^{-x_i} \circ \theta^{x_n} g_n^{-1}(t) \\ &= \theta^{-x'_n} \theta^{x'_n} g_n \theta^{-x'_n} \bigcirc_{i=n+1}^{n+k-1} \theta^{x'_i} g_i \theta^{-x'_i} \circ \theta^{x'_{n+k}} \theta^{-x_{n+k}} \bigcirc_{i=n+k-1}^{n+1} \theta^{x_i} g_i^{-1} \theta^{-x_i} \circ \theta^{x_n} g_n^{-1}(t) \\ &= g_n \theta^{-x_n} \bigcirc_{i=n+1}^{n+k-1} \theta^{x'_i} g_i \theta^{-x'_i} \circ \theta^{x'_{n+k}} \theta^{-x_{n+k}} \bigcirc_{i=n+k-1}^{n+1} \theta^{x_i} g_i^{-1} \theta^{-x_i} \circ \theta^{x_n} g_n^{-1}(t) \end{aligned}$$

Now re-ordering the terms, we get:

$$c = \bigcirc_{i=n}^{n+k-1} g_i \theta^{x'_{i+1}-x'_i} \circ \bigcirc_{i=n+k-1}^{n} \theta^{-(x_{i+1}-x_i)} g_i^{-1}(t).$$

At this point we see that c is of the form that we can apply Proposition 4.17 to $d(t,c)$. Thus:

$$\begin{aligned}
d(t,c) &= d\left(t, \bigcirc_{i=n}^{n+k-1} g_i \theta^{x'_{i+1}-x'_i} \circ \bigcirc_{i=n+k-1}^{n} \theta^{-(x_{i+1}-x_i)} g_i^{-1}(t)\right) \\
&\leqslant d\left(t, \bigcirc_{i=n+1}^{n+k} \theta^{(x'_{i+1}-x'_i)-(x_{i+1}-x_i)}(t)\right) + 4\sum_{i=n}^{n+k} \lambda_i \\
&= d\left(t, \bigcirc_{i=n+1}^{n+k} \theta^{(x'_{i+1}-x_{i+1})-(x'_i-x_i)}(t)\right) + 4\sum_{i=n}^{n+k} \lambda_i \\
&= d\left(t, \bigcirc_{i=n+1}^{n+k} \theta^{y_{i+1}}(t)\right) + 4\sum_{i=n}^{n+k} \lambda_i.
\end{aligned}$$

Now, $\bigcirc_{i=n+1}^{n+k} \theta^{y_{i+1}} = \theta^s$ where $s = \sum_{i=n+1}^{n+k} y_{i+1}$. But an easy calculation shows that in fact $s = (x'_{n+k+1} - x_{n+k+1}) - (x'_{n+1} - x_{n+1})$, so we obtain

$$d(t,c) \leqslant d\left(t, \theta^{(x'_{n+k+1}-x_{n+k+1})-(x'_{n+1}-x_{n+1})}(t)\right) + 4\sum_{i=n}^{n+k} \lambda_i,$$

so applying (3) we get

$$\begin{aligned}
d(t,c) &\leqslant (x'_{n+k+1} - x_{n+k+1}) - (x'_{n+1} - x_{n+1}) + 4\sum_{i=n}^{n+k} \lambda_i \\
&\leqslant |x'_{n+k+1} - x_{n+k+1}| + |x'_{n+1} - x_{n+1}| + 4\sum_{i=n}^{n+k} \lambda_i \\
&\leqslant 2\sup_{m\geqslant n} |x'_m - x_m| + 4\sum_{i=n}^{n+k} \lambda_i \qquad (21)
\end{aligned}$$

Now combining (19), (20), the fact that $c = \theta^{-x'_n} b$, (21) and (b) we get:

$$\begin{aligned}
d(H_n(x), H_{n+k}(x)) &\leqslant 2 \cdot 5 \sup_{m\geqslant n} |x'_m - x_m| + 4 \cdot 5 \sum_{i=n}^{n+k} \lambda_i \\
&\leqslant 10 \sup_{m\geqslant n} |x'_m - x_m| + 20 \sum_{i=n}^{\infty} \lambda_i \\
&\leqslant 10 \sup_{m\geqslant n} |x'_m - x_m| + 20\varepsilon_n. \qquad \square
\end{aligned}$$

4.19 Proposition. *Fix n and $k \geqslant 1$ and let $x \in \operatorname{Im} \hat{g}_{n+k-1}$. Then*

$$d(H_n(x), H_{n+k}(x)) \leqslant 15 \sup_{m \geqslant n} |x'_m - x_m| + 50\varepsilon_n.$$

Proof. Let $y \in T_{\varepsilon_{n+k}}$ be such that $\hat{g}_{n+k-1}(y) = x$ and let $y^* \in \operatorname{Im}(\theta^{x_{n+k}} g_{n+k} \theta^{-x_{n+k}})$ with $d(y, y^*) \leqslant \varepsilon_{n+k}$. Denote $x^* = \hat{g}_{n+k-1}(y^*)$. Now $x^* \in \operatorname{Im}(\hat{g}_{n+k-1} \circ \theta^{x_{n+k}} g_{n+k} \theta^{-x_{n+k}}) = \operatorname{Im}(\hat{g}_{n+k})$. By Proposition 4.11 we have that $d(x^*, x) \leqslant 5\varepsilon_{n+k}$. Let $z^* = \hat{g}_{n-1}^{-1}(x^*)$ and $z = \hat{g}_{n-1}^{-1}(x)$. Now

$$\begin{aligned} z^* &= \hat{g}_{n-1}^{-1} \hat{g}_{n+k-1}(y^*) \\ &= \bigcirc_{k=n-1}^{0} \theta^{x_k} g_k^{-1} \theta^{-x_k} \circ \bigcirc_{k=0}^{n+k-1} \theta^{x_k} g_k^{-1} \theta^{-x_k}(y^*) \\ &= \bigcirc_{k=n}^{n+k-1} \theta^{x_k} g_k \theta^{-x_k}(y^*). \end{aligned}$$

Similarly

$$z = \bigcirc_{k=n}^{n+k-1} \theta^{x_k} g_k^{-1} \theta^{-x_k}(y).$$

So we have $d(z, z^*) = d(h(y), h(y^*))$ where

$$h = \bigcirc_{k=n}^{n+k-1} \theta^{x_k} g_k \theta^{-x_k}.$$

By the same argument as in the proof of Proposition 4.11 we have that h is 5-Lipschitz, so

$$d(z, z^*) \leqslant 5d(y, y^*) \leqslant 5\varepsilon_{n+k}.$$

Additionally from (4) we have

$$d(H_{\varepsilon_{n+k}}^{x'_{n+k} - x_{n+k}}(y), H_{\varepsilon_{n+k}}^{x'_{n+k} - x_{n+k}}(y^*)) \leqslant d(y, y^*) + |x'_{n+k} - x_{n+k}|.$$

Now from triangle inequality we get:

$$d(H_n(x), H_{n+k}(x)) \leqslant d(H_n(x), H_n(x^*)) + d(H_n(x^*), H_{n+k}(x^*)) + d(H_{n+k}(x^*), H_{n+k}(x))$$

Let us consider the three terms separately. The first term by (17) and Proposition 4.11:

$$\begin{aligned} d(H_n(x), H_n(x^*)) &\leqslant d(\hat{g}'_{n-1} \theta^{x'_n - x_n} \hat{g}_{n-1}^{-1}(x), \hat{g}'_{n-1} \theta^{x'_n - x_n} \hat{g}_{n-1}^{-1}(x^*)) \\ &= d(\hat{g}'_{n-1} \theta^{x'_n - x_n}(z), \hat{g}'_{n-1} \theta^{x'_n - x_n}(z^*)) \\ &\leqslant 5d(\theta^{x'_n - x_n}(z), \theta^{x'_n - x_n}(z^*)) \\ &= 5d(z, z^*) \\ &\leqslant 25\varepsilon_{n+k} \\ &\leqslant 25\varepsilon_n. \end{aligned}$$

The second term using Proposition 4.18:

$$d(H_n(x^*), H_{n+k}(x^*)) \leqslant 10 \sup_{m \geqslant n} |x'_m - x_m| + 20\varepsilon_n.$$

And the third term using (18), (4), Proposition 4.11 and the facts that $y = \hat{g}^{-1}_{n+k-1}(x)$, $y^* = \hat{g}^{-1}_{n+k-1}(x^*)$ and $d(y, y^*) < \varepsilon_{n+k}$:

$$\begin{array}{rcl} d(H_{n+k}(x^*), H_{n+k}(x)) & \leqslant & d(\hat{g}'_{n+k-1} H^{x'_{n+k}-x_{n+k}}_{\varepsilon_{n+k}} \hat{g}^{-1}_{n+k-1}(x), \hat{g}'_{n+k-1} H^{x'_{n+k}-x_{n+k}}_{\varepsilon_{n+k}} \hat{g}^{-1}_{n+k-1}(x^*)) \\ & = & d(\hat{g}'_{n+k-1} H^{x'_{n+k}-x_{n+k}}_{\varepsilon_{n+k}}(y), \hat{g}'_{n+k-1} H^{x'_{n+k}-x_{n+k}}_{\varepsilon_{n+k}}(y^*)) \\ & \leqslant & 5d(H^{x'_{n+k}-x_{n+k}}_{\varepsilon_{n+k}}(y), H^{x'_{n+k}-x_{n+k}}_{\varepsilon_{n+k}}(y^*)) \\ & \leqslant & 5d(y, y^*) + 5|x'_{n+k} - x_{n+k}| \\ & \leqslant & 5\varepsilon_{n+k} + 5|x'_{n+k} - x_{n+k}| \\ & \leqslant & 5\varepsilon_n + 5 \sup_{m > n} |x'_m - x_m|. \end{array}$$

Combining these we get:

$$d(H_n(x), H_{n+k}(x)) \leqslant 15 \sup_{m \geqslant n} |x'_m - x_m| + 50\varepsilon_n. \qquad \square$$

By the definition of H_n, for all $x \notin \bigcap_{n \in \mathbb{N}} \operatorname{Im}(\hat{g}_n)$ there is m such that for all $n > m$ we have $H_m(x) = H_n(x)$. In fact if n is such that $x \notin \operatorname{Im} \hat{g}_n$, then $H_m(x) = H_n(x)$ for all $m > n$. Thus the point-wise limit $H_\infty = \lim_{n \to \infty} H_n$ is well defined on the complement of $K(\bar{x})$ for which we have

$$x \notin \operatorname{Im} \hat{g}_n \Rightarrow H_\infty(x) = H_n(x) \tag{22}$$

H_∞ is a homeomorphism taking the complement of $K(\bar{x})$ to the complement of $K(\bar{x}')$.

Suppose $x \in \operatorname{Im} \hat{g}_{n-1} \setminus \operatorname{Im} \hat{g}_n$ and $z \in \operatorname{Im} \hat{g}_{n+k-1} \setminus \operatorname{Im} \hat{g}_{n+k}$, $k \geqslant 1$. Now from (22) and Propositions 4.15 and 4.19 we have:

$$\begin{array}{rcl} d(H_\infty(x), H_\infty(z)) & = & d(H_n(x), H_{n+k}(z)) \\ & \leqslant & d(H_n(x), H_n(z)) + d(H_n(z), H_{n+k}(z)) \\ & \leqslant & 25d^{p(n-1)}(x, z) + 5|x'_n - x_n| + 15 \sup_{m \geqslant n} |x'_m - x_m| + 50\varepsilon_n. \\ & \leqslant & 25d^{p(n-1)}(x, z) + 20 \sup_{m \geqslant n} |x'_m - x_m| + 50\varepsilon_n. \end{array}$$

The latter is independent of k, so (changing $n-1$ to n) we have for all $x \in \operatorname{Im} \hat{g}_n \setminus \operatorname{Im} \hat{g}_{n+1}$ and $z \in \operatorname{Im} \hat{g}_{n+1}$ that:

$$d(H_\infty(x), H_\infty(z)) \leqslant 100(d^{p(n)}(x, z) + \sup_{m > n} |x'_m - x_m| + \varepsilon_{n+1}). \tag{23}$$

4.20 Proposition. *If (z_n) is a Cauchy sequence with $z_n \in \operatorname{Im} \hat{g}_n$ then for every ε there is n such that for all $m > n$ we have $d^{p(n)}(z_n, z_m) < \varepsilon$.*

Proof. If not, pick an ε and a subsequence (z_{n_k}) so that for all k we have $d^{p(n(k))}(z_{n(k)}, z_{n(k+1)}) \geqslant \varepsilon$ where $n_k = n(k)$. Fix k so large that $5\varepsilon_{n(k)-1} < \varepsilon$. Now for all k' we have $d^{p(n(k'))}(z_{n(k')}, z_{n(k'+1)}) \geqslant \varepsilon \geqslant 5\varepsilon_{n(k)-1}$. Now we can pick $k' > k$ to be so big that $d(z_{n(k')}, z_{n(k'+1)}) < \varepsilon_{n(k)}$. By Proposition 4.12, we have

$$d^{p(n(k'))}(z_{n(k')}, z_{n(k'+1)}) \leqslant 5d(\hat{g}_{n(k)}^{-1}(z_{n(k')}), \hat{g}_{n(k)}^{-1}(z_{n(k'+1)})).$$

So we have

$$\begin{aligned} 5\varepsilon_{n(k)-1} &\leqslant d^{p(n(k'))}(z_{n(k')}, z_{n(k'+1)}) \\ &\leqslant 5d(\hat{g}_{n(k)}^{-1}(z_{n(k')}), \hat{g}_{n(k)}^{-1} z_{n(k'+1)}) \\ \iff \varepsilon_{n(k)-1} &\leqslant d(\hat{g}_{n(k)}^{-1} z_{n(k')}, \hat{g}_{n(k)}^{-1} z_{n(k'+1)}) \end{aligned}$$

And because $\hat{g}_{n(k)}$ is $\varepsilon_{n(k)-1}/\varepsilon_{n(k)}$-colipschitz (cf. (11)),

$$d(z_{n(k)}, z_{n(k')}) \geqslant \varepsilon_{n(k)}/\varepsilon_{n(k)-1} d(\hat{g}_{n(k)}^{-1} z_{n(k')}, \hat{g}_{n(k)}^{-1} z_{n(k'+1)}) \geqslant \varepsilon_{n(k)}$$

which is a contradiction with the choice of k'. □

4.21 Proposition. *If $\lim_{n\to\infty}(x'_n - x_n) = 0$, then the knots $K(\bar{x})$ and $K(\bar{x}')$ are equivalent.*

Proof. It remains to show that H_∞ which is a homeomorphism from the complement of $K(\bar{x})$ to the complement of $K(\bar{x}')$ extends to the knots themselves. It will follow if we show that a sequence (z_n) in the complement of $K(\bar{x})$ is Cauchy if and only if $(H_\infty(z_n))$ is Cauchy in the complement of $K(\bar{x}')$. The "if" part will follow by symmetry once we prove the "only if" part.

So suppose (z_n) is Cauchy in the complement of $K(\bar{x})$ and fix ε. For each n let $k(n)$ be the largest number such that $z_n \in \operatorname{Im} \hat{g}_{k(n)}$. Without loss of generality assume that $k(n)$ is strictly increasing in n. Choose n so large that for all $m > n$, $100 d^{p(n)}(z_n, z_m) < \varepsilon/3$ which exists by Proposition 4.20, so that $100 \sup_{m \geqslant n+1} |x'_m - x_m| < \varepsilon/3$ which is possible by the convergence, and so that $100\varepsilon_{n+1} < \varepsilon/3$ which is possible by (b). Since $z_n \in \operatorname{Im} \hat{g}_{k(n)} \setminus \operatorname{Im} \hat{g}_{k(n)+1}$ and $z_m \in \hat{g}_{k(n)+1}$ for $m > n$, by inequality (23) we have for all $m > n$

$$\begin{aligned} d(H_\infty(z_n), H_\infty(z_m)) &\leqslant 100(d^{p(k(n))}(z_n, z_m) + \sup_{m \geqslant k(n)+1} |x'_m - x_m| + \varepsilon_{k(n)+1}) \\ &\leqslant 100(d^{p(n)}(z_n, z_m) + \sup_{m \geqslant n+1} |x'_m - x_m| + \varepsilon_{n+1}) \\ &\leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

The second inequality follows simply from the fact that $k(n) \geqslant n$. □

Recall the definition of $f^{\bar{x}}$, just prior to Proposition 4.13.

4.22 Proposition. *Suppose $(x_{n(k)})_k$ is a convergent subsequence of (x_n) which converges to some a. Let $\varepsilon > 0$ be arbitrarily small and let γ be the connected component of $f^{\bar{x}}[S^1] \cap B(f^{\bar{x}}(a), \varepsilon)$ with $f^{\bar{x}}(a) \in \gamma$. Then there exists k such that for all $k' > k$, $(B(f^{\bar{x}}(a), \varepsilon), \gamma)$ contains $K_{n(k')}$ strongly as a component. (Recall Definition 4.5.)*

Proof. For simplicity of notation denote $n_k = n(k)$. Let k be so big that $5(d(x_{n(k)}, a)+\lambda_{n(k)}) < \varepsilon/2$. Let $k' > k$. Now

$$\theta^{x_{n(k')}} f^{\bar{x}}_{n(k')} \theta^{-x_{n(k')}}$$

has $K_{n(k')}$ (strongly) as a component in $B(x_{n(k')}, \lambda_{n(k')})$ by (f) and so

$$\hat{f}^{\bar{x}}_{n(k')} = \hat{g}^{\bar{x}}_{n-1} \theta^{x_{n(k')}} f^{\bar{x}}_{n(k')} \theta^{-x_{n(k')}}$$

contains $K_{n(k')}$ strongly as a component inside

$$\hat{g}^{\bar{x}}_{n-1}[B(x_{n(k')}, \lambda_{n(k')})]$$

and this is witnessed by $20\varepsilon_{n+1}$ by (c) and Lemmas 4.4 and 4.6. On the other hand $d(f^{\bar{x}}, \hat{f}^{\bar{x}}_{n(k')}) < 10\varepsilon_{n+1} \leqslant (20\varepsilon_{n+1})/2$ which by Lemma 4.7 means that $f^{\bar{x}}$ has $K_{n(k')}$ as a component in

$$\hat{g}^{\bar{x}}_{n-1}[B(x_{n(k')}, \lambda_{n(k')})]$$

as well. On the other hand, by the choice of k,

$$\hat{g}^{\bar{x}}_{n-1}[B(x_{n(k')}, \lambda_{n(k')})] \subset B(f^{\bar{x}}(a), \varepsilon). \qquad \square$$

4.23 Proposition. *Suppose $a, a' \in S^1$, $n \in \mathbb{N}$ and $B \subset S^3$ are such that B is a set homeomorphic to a closed ball with $f^{\bar{x}}(a)$ in the interior and $f^{\bar{x}}(a')$ in the exterior. Suppose $(x_{n(k)})$ is a subsequence converging to a'. Let γ be the connected component of $f^{\bar{x}} \cap B$ which contains $f^{\bar{x}}(a)$. Then there exists k such that for all $k' > k$, (B, γ) does not contain $K_{n(k')}$ strongly as a component.*

Proof. Since B is closed, there is B', also homeomorphic to a closed ball which contains $f^{\bar{x}}(x_n)$ in the interior and disjoint from B. Let γ' be the connected component of $f^{\bar{x}} \cap B'$ which contains $f^{\bar{x}}(a')$. Then Let k be as given by Proposition 4.22, i.e. for all $k' > k$, (B', γ') contains $K_{n(k')}$ strongly as a component. If now also (B, γ) contains $K_{n(k')}$ strongly as a component, then it is easy to see from the definition of strong components that $f^{\bar{x}}$ contains $K_{n(k')} \# K_{n(k')}$ strongly as a component. But this is impossible because $d(f^{\bar{x}}, \hat{f}^{\bar{x}}_m) \overset{m\to\infty}{\longrightarrow} 0$ and $\hat{f}^{\bar{x}}_m$ does not contain $K_{n(k')} \# K_{n(k')}$ strongly as a component for any m. $\square$

4.24 Proposition. *Suppose (x'_n) and (x_n) are such that there exists $A \subset \mathbb{N}$ so that for all $n \in A$ $x'_n = x_n$ and $\{x'_n \mid n \in A\} = \{x_n \mid n \in A\}$ is dense. Then, if $\lim_{n\to\infty}(x'_n - x_n)$ does not exist, then the knots $K(\bar{x})$ and $K(\bar{x}')$ are not equivalent.*

Proof. The set $\{(x'_n - x_n) \mid n \notin A\}$ must have at least one cluster point by the compactness of S^1 and at least one cluster point that is not equal to 0, because otherwise $(x'_n - x_n)$ would converge to zero. Let $(n(k))_k$ be an increasing sequence of natural numbers so that $z = \lim_{k\to\infty}(x'_{n(k)} - x_{n(k)})$ exists and $z \neq 0$. Further, take a subsequence of this $(n(k(j)))_j$ such that $(x_{n(k(j))})_j$ and $(x'_{n(k(j))})_j$ converge to points $a, a' \in S^1$ respectively. These points must be distinct, in fact $a' - a = z$. We will now arrive at a contradiction from the assumption that there is a homeomorphism $h\colon S^3 \to S^3$ with $h[K(\bar{x})] = K(\bar{x}')$. There are two cases:

Case 1 $h(f^{\bar{x}}(a)) = f^{\bar{x}'}(a')$. Let ε be so small that

$$B(f^{\bar{x}}(a), \varepsilon) \cap B(f^{\bar{x}}(a'), \varepsilon) = \varnothing$$

and

$$B(f^{\bar{x}'}(a), \varepsilon) \cap h[B(f^{\bar{x}}(a), \varepsilon)] = \varnothing.$$

The last is possible because $h(f^{\bar{x}}(a)) = f^{\bar{x}'}(a') \neq f^{\bar{x}'}(a)$. Since $\{x_m \mid m \in A\}$ is dense, pick a subsequence $(x_{m(k)})$ converging to a. Let γ and γ' be the connected components of $f^{\bar{x}} \cap B(f^{\bar{x}}(a), \varepsilon)$ and $f^{\bar{x}'} \cap h[B(f^{\bar{x}}(a), \varepsilon)]$ containing $f^{\bar{x}}(a)$ and $f^{\bar{x}'}(a')$ respectively. Note that in fact $h[\gamma] = \gamma'$. By Proposition 4.22 there is $k \in A$ such that $(B(f^{\bar{x}}(a), \varepsilon), \gamma)$ has $K_{m(k_1)}$ strongly as a component for all $k_1 > k$ and by Proposition 4.23 there is k' such that $(h[B(f^{\bar{x}}(a), \varepsilon)], \gamma')$ does not have $K_{m(k_2)}$ strongly as a component for any $k_2 > k'$. By taking $k_3 > \max\{k, k'\}$ this contradicts Lemma 4.6.

Case 2 $h(f^{\bar{x}}(a)) \neq f^{\bar{x}'}(a')$. This is similar to Case 1, but it is also presented here for the sake of completeness. Let ε be so small that

$$B(f^{\bar{x}}(a), \varepsilon) \cap B(f^{\bar{x}}(a'), \varepsilon) = \varnothing$$

and

$$B(f^{\bar{x}'}(a'), \varepsilon) \cap h[B(f^{\bar{x}}(a), \varepsilon)] = \varnothing.$$

Let γ and γ' be the connected components of $f^{\bar{x}} \cap B(f^{\bar{x}}(a), \varepsilon)$ and $f^{\bar{x}'} \cap h[B(f^{\bar{x}}(a), \varepsilon)]$ containing $f^{\bar{x}}(a)$ and $f^{\bar{x}'}(a')$ respectively. By the above, $(x_{n(k(j))})$ converges to a, so by Proposition 4.22 there exists j_0 such that for all $j > j_0$, $(B(f^{\bar{x}}(a), \varepsilon), \gamma)$ contains $K_{n(k(j))}$ strongly as a component. On the other hand $(x'_{n(k(j))})$ converges to a' which means by Proposition 4.23 that there is j_1 such that for all $j > j_1$ $(h[B(f^{\bar{x}}(a), \varepsilon)], \gamma')$ does *not* contain $K_{n(k(j))}$ strongly as a component. This is again a contradiction with Lemma 4.6. □

Now we are ready to prove the main theorem.

Proof of Theorem 4.1. Fix a dense $(z_n) \subset S^1$. Let $(x_n) \in (S^1)^{\mathbb{N}}$ and define (y_n) by $y_{2k} = z_k$ and $y_{2k+1} = x_k$ for all k. Then let $F(\bar{x}) = K(\bar{y})$ as defined by (7). Suppose (x_n) and (x'_n) are E^*-equivalent sequences. Let (y_n) and (y'_n) be the corresponding sequences as above. Now $(x'_n - x_n) \xrightarrow{n\to\infty} 0$ and of course also $(y'_n - y_n) \xrightarrow{n\to\infty} 0$. So by

Proposition 4.21 $K(\bar{y}'_n)$ and $K(\bar{y}_n)$ are equivalent. Suppose that $(x'_n - x_n)$ does not converge to zero. Then $(y'_n - y_n)$ does not converge at all. Taking the even numbers as A, Proposition 4.24 implies that $K(\bar{y}'_n)$ and $K(\bar{y}_n)$ are not equivalent.

Let us show that the reduction is continuous. Let $\bar{x}$ be a sequence and $\varepsilon > 0$ and let us find a neighborhood U of $\bar{x}$ such that for all $\bar{x}' \in U$ we have $d(K(\bar{x}), K(\bar{x}')) < \varepsilon$. Let k be so big that $21\varepsilon_k < \varepsilon$. For $n \leqslant k$ let Let

$$\delta_n = \frac{\varepsilon_k}{3 \cdot 2^n (k+1)}. \tag{24}$$

Let U be the neighborhood

$$U = \prod_{n=0}^{k} B_{S^1}(x_n, \delta_n) \times \prod_{n>k} S^1.$$

For $m < n$ denote $\hat{g}^{\bar{x}}_{m,n} = \bigcirc_{i=m}^{n} \theta^{x_i} g_i \theta^{-x_i}(z)$ (cf. 6). Now for all $z \in T_{\varepsilon_{k+1}}$ and $\bar{x}' \in U$ we have

$$\begin{aligned}
d(\hat{g}^{\bar{x}}_k(z), \hat{g}^{\bar{x}'}_k(z)) &= d\big((\theta^{x_0} g_0 \theta^{-x_0} \circ \hat{g}^{\bar{x}}_{1,k})(z), (\theta^{x'_0} g_0 \theta^{-x'_0} \circ \hat{g}^{\bar{x}'}_{1,k})(z)\big) \\
&= d\big((\theta^{x'_0 - x_0} g_0 \theta^{-x_0} \circ \hat{g}^{\bar{x}}_{1,k})(z), (g_0 \theta^{-x'_0} \circ \hat{g}^{\bar{x}'}_{1,k})(z)\big) \\
&\leqslant |x'_0 - x_0| + d\big((g_0 \theta^{-x_0} \circ \hat{g}^{\bar{x}}_{1,k})(z), (g_0 \theta^{-x'_0} \circ \hat{g}^{\bar{x}'}_{1,k})(z)\big).
\end{aligned}$$

The last inequality holds for a similar reason as (4). The first equality is by the definition (6) and the second follows from that θ^s is an isometry. Continuing the chain of inequalities, using the facts that $|x'_0 - x_0| < \delta_0$ and g_0 is L_0-Lipschitz and $L_0 < 2$, (by (b), (h), Proposition 4.2 and the discussion after it), we have

$$\begin{aligned}
&\leqslant \delta_0 + 2d\big((\theta^{-x_0} \circ \hat{g}^{\bar{x}}_{1,k})(z), (\theta^{-x'_0} \circ \hat{g}^{\bar{x}'}_{1,k})(z)\big) \\
&= \delta_0 + 2d\big((\theta^{x'_0 - x_0} \circ \hat{g}^{\bar{x}}_{1,k})(z), \hat{g}^{\bar{x}'}_{1,k}(z)\big) \\
&\leqslant \delta_0 + 2(\delta_0 + d\big(\hat{g}^{\bar{x}}_{1,k}(z)\, \hat{g}^{\bar{x}'}_{1,k}(z)\big)) \\
&= 3\delta_0 + 2d\big(\hat{g}^{\bar{x}}_{1,k}(z), \hat{g}^{\bar{x}'}_{1,k}(z)\big)\big).
\end{aligned}$$

Continuing by induction in the exact same way we get

$$\begin{aligned}
&= 3\delta_0 + 2d(\hat{g}^{\bar{x}}_{1,k}(z), \hat{g}^{\bar{x}'}_{1,k}(z))) \\
&\leqslant 3\delta_0 + 2(3\delta_1 + 2d(\hat{g}^{\bar{x}}_{2,k}(z), \hat{g}^{\bar{x}'}_{2,k}(z))) \\
&\leqslant 3\delta_0 + 3 \cdot 2\delta_1 + 2^2 d(\hat{g}^{\bar{x}}_{2,k}(z), \hat{g}^{\bar{x}'}_{2,k}(z))) \\
&\vdots \quad \vdots \\
&\leqslant 3\delta_0 + 3 \cdot 2\delta_1 + 3 \cdot 2^2 \delta_2 + \cdots + 3 \cdot 2^k \delta_k \\
&= \sum_{n=0}^{k} 3 \cdot 2^n \delta_n.
\end{aligned}$$

By the definition of δ_n, (24), this equals to

$$\sum_{n=0}^{k} \frac{\varepsilon_k}{k+1} = \varepsilon_k.$$

Thus, we get that for all $z \in T_{\varepsilon_{k+1}}$ and $\bar{x}' \in U$ we have

$$d(\hat{g}_k^{\bar{x}}(z), \hat{g}_k^{\bar{x}'}(z)) \leqslant \varepsilon_k$$

and in particular for all $s \in S^1$ and and $\bar{x}' \in U$ we have

$$d(\hat{f}_k^{\bar{x}}(s), \hat{f}_k^{\bar{x}'}(s)) \leqslant \varepsilon_k.$$

From the proof of Proposition 4.13, we know that in the sup-metric

$$d(\hat{f}_k^{\bar{x}}, f^{\bar{x}}) \leqslant 10\varepsilon_k \quad \text{and} \quad d(\hat{f}_k^{\bar{x}'}, f^{\bar{x}'}) \leqslant 10\varepsilon_k$$

so from the above and the choice of k, we have

$$d(f^{\bar{x}}, f^{\bar{x}'}) \leqslant 21\varepsilon_k < \varepsilon$$

in the sup metric. But this of course implies that the Hausdorff distance of $K(\bar{x}) = \mathrm{Im}(f^{\bar{x}})$ and $K(\bar{x}') = \mathrm{Im}(f^{\bar{x}'})$ is at most ε as well. Since the function $\bar{x} \mapsto \bar{y}$ in the definition of $F(\bar{x})$ is continuous on $(S^1)^{\mathbb{N}}$, the reduction F is continuous. □

Remark. As remarked after (22), all knots in the above construction have homeomorphic complements. This strengthens the side remark of [Nan14] that there are uncountably many non-equivalent knots with homeomorphic complements to that there is a *non-classifiable* uncountable set of knots all of which have homeomorphic complements.

References

[DH67] P. H. Doyle and J. G. Hocking. A generalization of the wilder arcs. *Pacific Journal of Mathematics*, 22(03):397–399, 1967.

[FH62] R. Fox and O. G. Harrold. The wilder arcs. *Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute)*, pages 184–187, 1962.

[Fri03] G. Friedman. Alexander polynomials of non-locally-flat knots. *Indiana Univ. Math. J.*, 52(06):1479–1578, 2003.

[Fri04] G. Friedman. Intersection Alexander polynomials. *Topology*, 43(1):71 – 117, 2004.

[Fri05] G. Friedman. Groups of locally-flat disk knots and non-locally-flat sphere knots. *Journal of Knot Theory and Its Ramifications*, 14(02):189–215, 2005.

[FS89] H. Friedman and L. Stanley. A Borel reducibility theory for classes of countable structures. *Journal of Symbolic Logic*, 54(3):894–914, 1989.

[Gao08] S. Gao. *Invariant Descriptive Set Theory*. Chapman & Hall/CRC Pure and Applied Mathematics. Taylor & Francis, 2008.

[GRWŽ11] D. Garity, D. Repovš, D. Wright, and M. Željko. Distinguishing Bing-Whitehead Cantor sets. *Transactions of the AMS*, 363:1007–1022, 2011.

[Hjo00] G. Hjorth. *Classification and Orbit Equivalence Relations*. Mathematical Surveys and Monographs. American Mathematical Society, 2000.

[JL09] J. Juyumaya and S. Lambropoulou. An invariant for singular knots. *Journal of Knot Theory and Its Ramifications*, 18(06):825–840, 2009.

[Kec94] A. S. Kechris. *Classical Descriptive Set Theory*, volume 156. Springer-Verlag, Graduate Texts in Mathematics, 1994.

[Lom64] S.J. Lomonaco. *An Algebraic Theory of Local Knottedness*. Princeton University, 1964.

[McP73] J. M. McPherson. Wild arcs in three-space. *Pacific Journal of Mathematics*, 45(02):585–598, 1973.

[Nan14] O. Nanyes. Invariants for everywhere wild knots. *Journal of Knot Theory and Its Ramifications*, 23(06), 2014.

[RN65] C. Ryll-Nardzewski. On a Freedman's problem. *Fundamenta Mathematicae*, 57(3):273–274, 1965.

[Ros11] C. Rosendal. Descriptive set theory and separable Banach spaces. *Notices of the AMS*, 58(9):1251–1262, 2011.